Passive mechanical synthesis and n-port resistive networks

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- Restricted complexity synthesis
- Passive mechanical control
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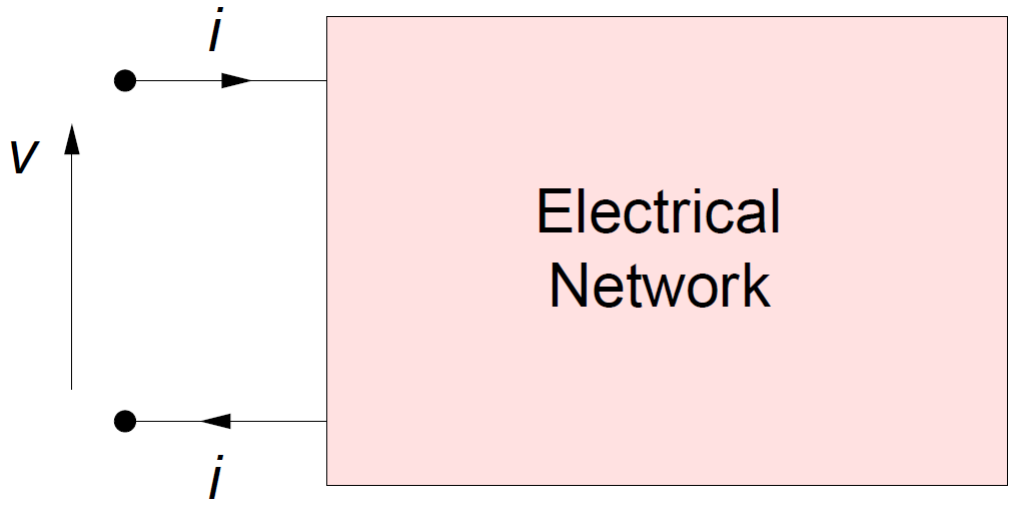
Before everyone knows

Raced by Kimi Raikonnen at the 2005 Spanish Grand Prix in Barcelona.

Raikonnen won the race to give McLaren their first victory of the 2005 season.

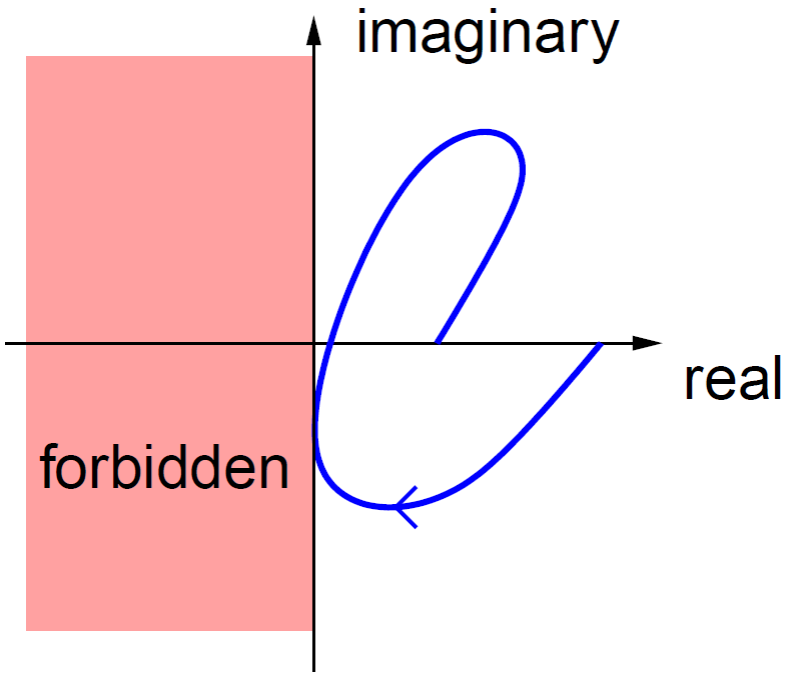
Classical Electrical Circuit Theory

Passivity is characterised in terms of terminal behaviour (impedance or admittance functions).



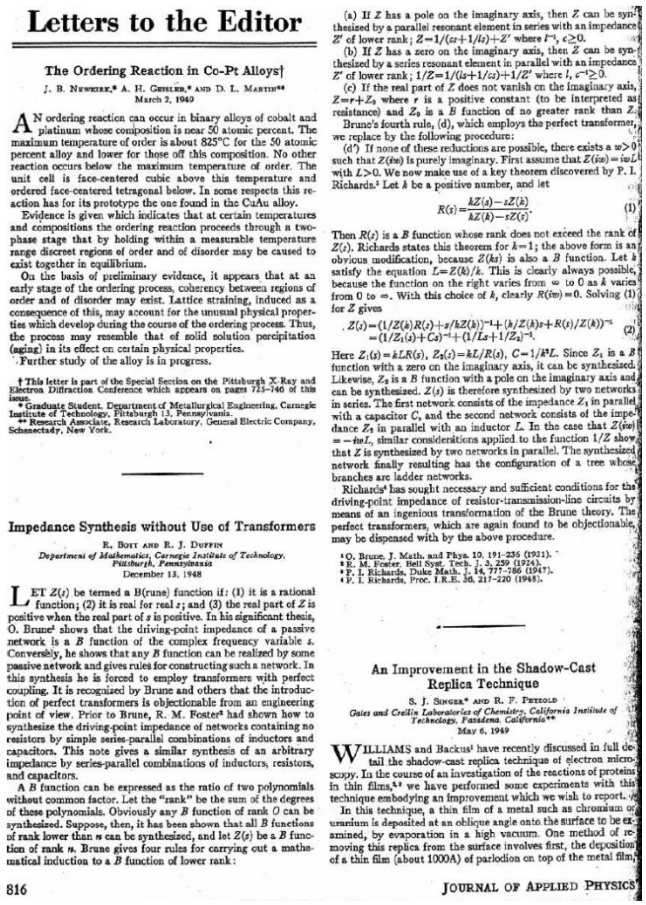
Admittance $= Y(s) = \hat{i}(s)/\hat{v}(s)$. Impedance $= Z(s) = Y(s)^{-1}$.

Positive-real Functions



The admittance (or impedance) of a passive network must be *positive-real*, i.e. $\mathrm{Re}(Y(j\omega)) \geq 0$ for all ω.

Bott-Duffin Synthesis



Letters to the Editor

The Ordering Reaction in Co-Pt Alloys†

J. B. Newkirk,* A. H. Geisler,* and D. L. Martin**
March 2, 1949

An ordering reaction can occur in binary alloys of cobalt and platinum whose composition is near 50 atomic percent. The maximum temperature of order is about 825°C for the 50 atomic percent alloy and lower for those off this composition. No other reaction occurs below the maximum temperature of order. The unit cell is face-centered cubic above this temperature and ordered face-centered tetragonal below. In some respects this reaction has for its prototype the one found in the CuAu alloy.

Evidence is given which indicates that at certain temperatures and compositions the ordering reaction proceeds through a two-phase stage that by holding within a measurable temperature range discret regions of order and of disorder may be caused to exist together in equilibrium.

On the basis of preliminary evidence, it appears that at an early stage of the ordering process, coherency between regions of order and of disorder may exist. Lattice straining, induced as a consequence of this, may account for the unusual physical properties which develop during the course of the ordering process. Thus, the process may resemble that of solid solution percipitation (aging) in its effect on certain physical properties.

Further study of the alloy is in progress.

† This letter is part of the Special Section on the Pittsburgh X-Ray and Electron Diffraction Conference which appears on pages 725-760 of this issue.

* Graduate Student, Department of Metallurgical Engineering, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.

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Impedance Synthesis without Use of Transformers

R. Bott and R. J. Duffin
Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
December 13, 1948

Let $Z(s)$ be termed a B(rune) function if: (1) it is a rational function; (2) it is real for real s; and (3) the real part of Z is positive when the real part of s is positive. In his significant thesis, O. Brune[1] shows that the driving-point impedance of a passive network is a B function of the complex frequency variable s. Conversely, he shows that any B function can be realized by some passive network and gives rules for constructing such a network. In this synthesis he is forced to employ transformers with perfect coupling. It is recognized by Brune and others that the introduction of perfect transformers is objectionable from an engineering point of view. Prior to Brune, R. M. Foster[2] had shown how to synthesize the driving-point impedance of networks containing no resistors by simple series-parallel combinations of inductors and capacitors. This note gives a similar synthesis of an arbitrary impedance by series-parallel combinations of inductors, resistors, and capacitors.

A B function can be expressed as the ratio of two polynomials without common factor. Let the "rank" be the sum of the degrees of these polynomials. Obviously any B function of rank 0 can be synthesized. Suppose, then, it has been shown that all B functions of rank lower than n can be synthesized, and let $Z(s)$ be a B function of rank n. Brune gives four rules for carrying out a mathematical induction to a B function of lower rank:

(a) If Z has a pole on the imaginary axis, then Z can be synthesized by a parallel resonant element in series with an impedance Z' of lower rank; $Z=1/(cs+1/ls)+Z'$ where l^{-1}, $c\geq 0$.

(b) If Z has a zero on the imaginary axis, then Z can be synthesized by a series resonant element in parallel with an impedance Z' of lower rank; $1/Z=1/(ls+1/cs)+1/Z'$ where l, $c^{-1}\geq 0$.

(c) If the real part of Z does not vanish on the imaginary axis, $Z=r+Z_1$ where r is a positive constant (to be interpreted as resistance) and Z_1 is a B function of no greater rank than Z.

Brune's fourth rule, (d), which employs the perfect transformer, we replace by the following procedure:

(d') If none of these reductions are possible, there exists a $w>0$ such that $Z(iw)$ is purely imaginary. First assume that $Z(iw)=iwL$ with $L>0$. We now make use of a key theorem discovered by P. I. Richards.[3] Let k be a positive number, and let

$$R(s)=\frac{kZ(s)-sZ(k)}{kZ(k)-sZ(s)}. \quad (1)$$

Then $R(s)$ is a B function whose rank does not exceed the rank of $Z(s)$. Richards states this theorem for $k=1$; the above form is an obvious modification, because $Z(ks)$ is also a B function. Let k satisfy the equation $L=Z(k)/k$. This is clearly always possible, because the function on the right varies from ∞ to 0 as k varies from 0 to ∞. With this choice of k, clearly $R(iw)=0$. Solving (1) for Z gives

$$Z(s)=(1/Z(k)R(s)+s/kZ(k))^{-1}+(k/Z(k)s+R(s)/Z(k))^{-1} \quad (2)$$
$$=(1/Z_1(s)+Cs)^{-1}+(1/Ls+1/Z_2)^{-1}.$$

Here $Z_1(s)=kLR(s)$, $Z_2(s)=kL/R(s)$, $C=1/k^2L$. Since Z_1 is a B function with a zero on the imaginary axis, it can be synthesized. Likewise, Z_2 is a B function with a pole on the imaginary axis and can be synthesized. $Z(s)$ is therefore synthesized by two networks in series. The first network consists of the impedance Z_1 in parallel with a capacitor C, and the second network consists of the impedance Z_2 in parallel with an inductor L. In the case that $Z(iw)=-iwL$, similar considerations applied to the function $1/Z$ show that Z is synthesized by two networks in parallel. The synthesized network finally resulting has the configuration of a tree whose branches are ladder networks.

Richards[4] has sought necessary and sufficient conditions for the driving-point impedance of resistor-transmission-line circuits by means of an ingenious transformation of the Brune theory. The perfect transformers, which are again found to be objectionable, may be dispensed with by the above procedure.

[1] O. Brune, J. Math. and Phys. 10, 191-236 (1931).
[2] R. M. Foster, Bell Syst. Tech. J. 3, 259 (1924).
[3] P. I. Richards, Duke Math. J. 14, 777-786 (1947).
[4] P. I. Richards, Proc. I.R.E. 36, 217-220 (1948).

An Improvement in the Shadow-Cast Replica Technique

S. J. Singer* and R. F. Preseld
Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena, California**
May 6, 1949

Williams and Backus[1] have recently discussed in full detail the shadow-cast replica technique of electron microscopy. In the course of an investigation of the reactions of proteins in thin films,[2,3] we have performed some experiments with this technique embodying an improvement which we wish to report. In this technique, a thin film of a metal such as chromium or uranium is deposited at an oblique angle onto the surface to be examined, by evaporation in a high vacuum. One method of removing this replica from the surface involves first, the deposition of a thin film (about 1000A) of parlodion on top of the metal film,

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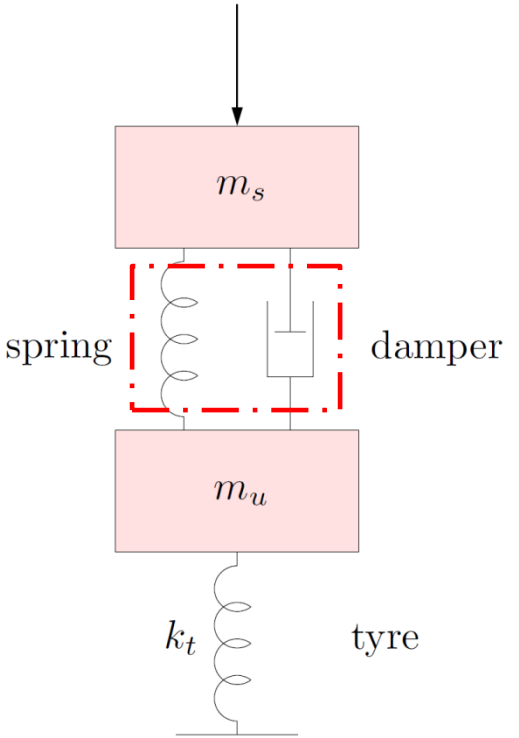
Journal of Applied Physics

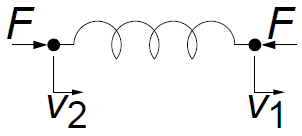
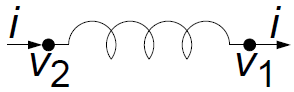
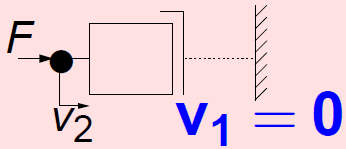
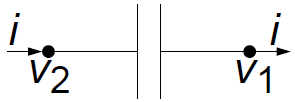
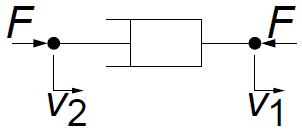
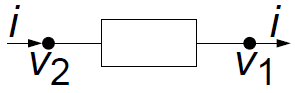
Positive Real Function → **Circuit Realisation**

constructive procedure



→ **Passive Black-box Mechanism**

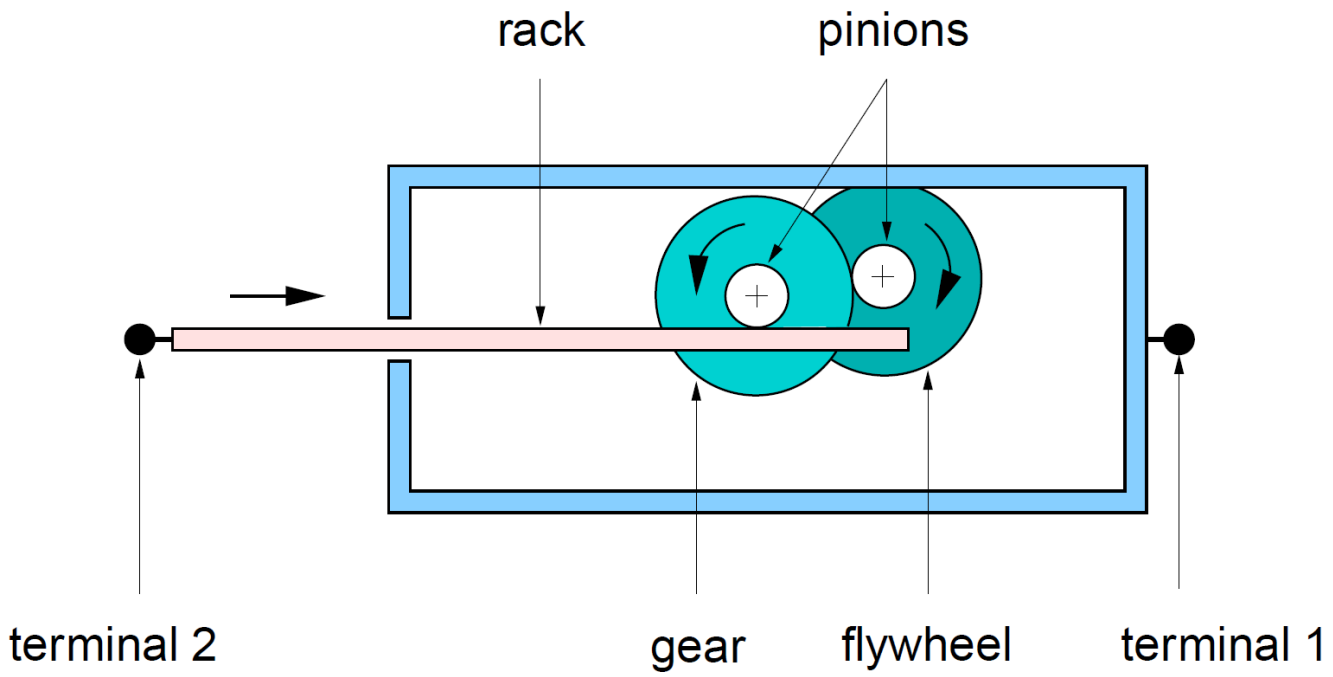
R. Bott and R.J. Duffin. Impedance synthesis without use of transformers. *J. Appl. Phys.*, 20:816, 1949.

Mechanical	Electrical
 spring	 inductor
 mass	 capacitor
 damper	 resistor

Mass element only has one terminal—a fundamental restriction for synthesis.

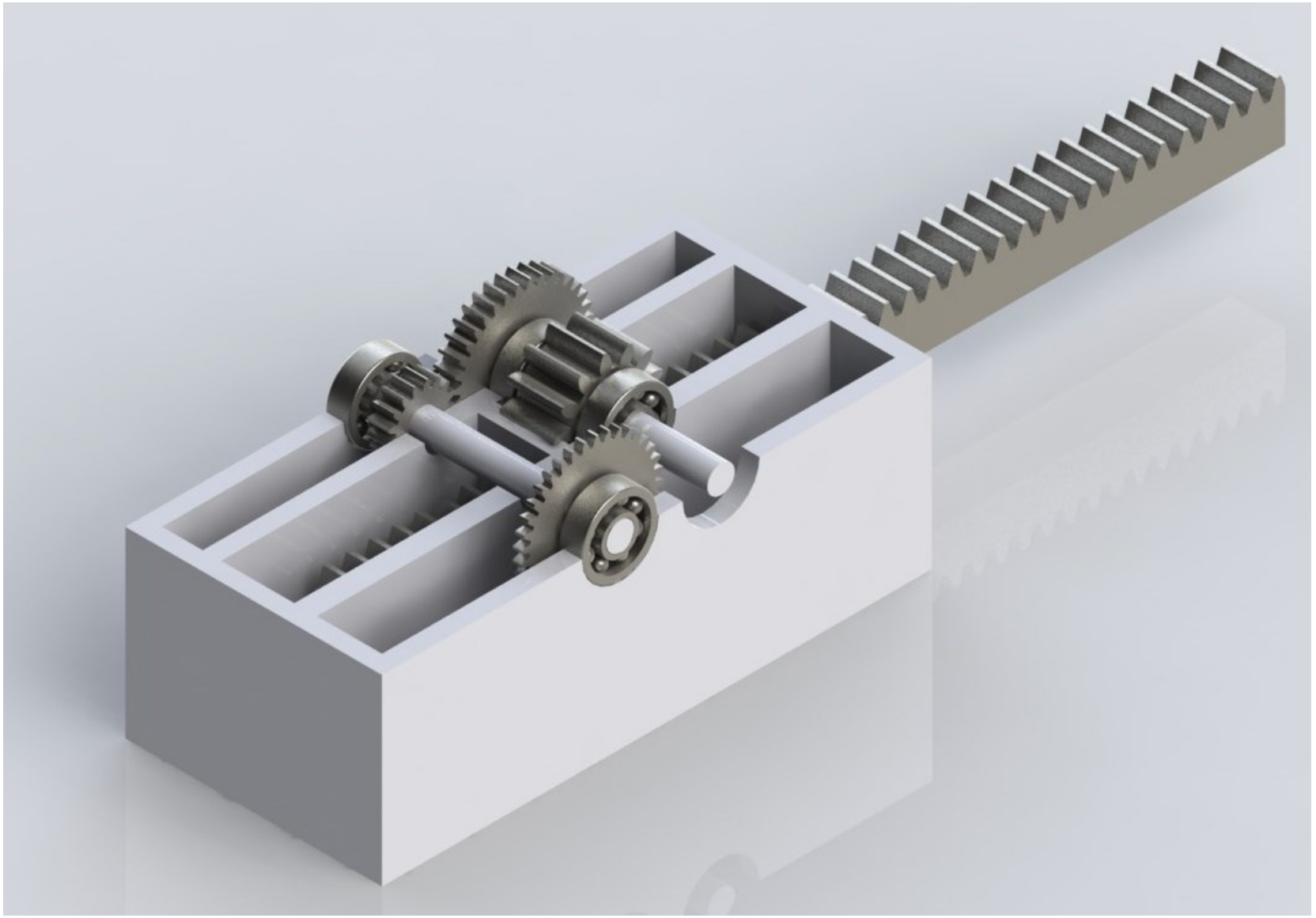
Is it possible to construct a physical device such that the relative acceleration between its endpoints is proportional to the applied force?

Yes！ The device is named “inerter” *.

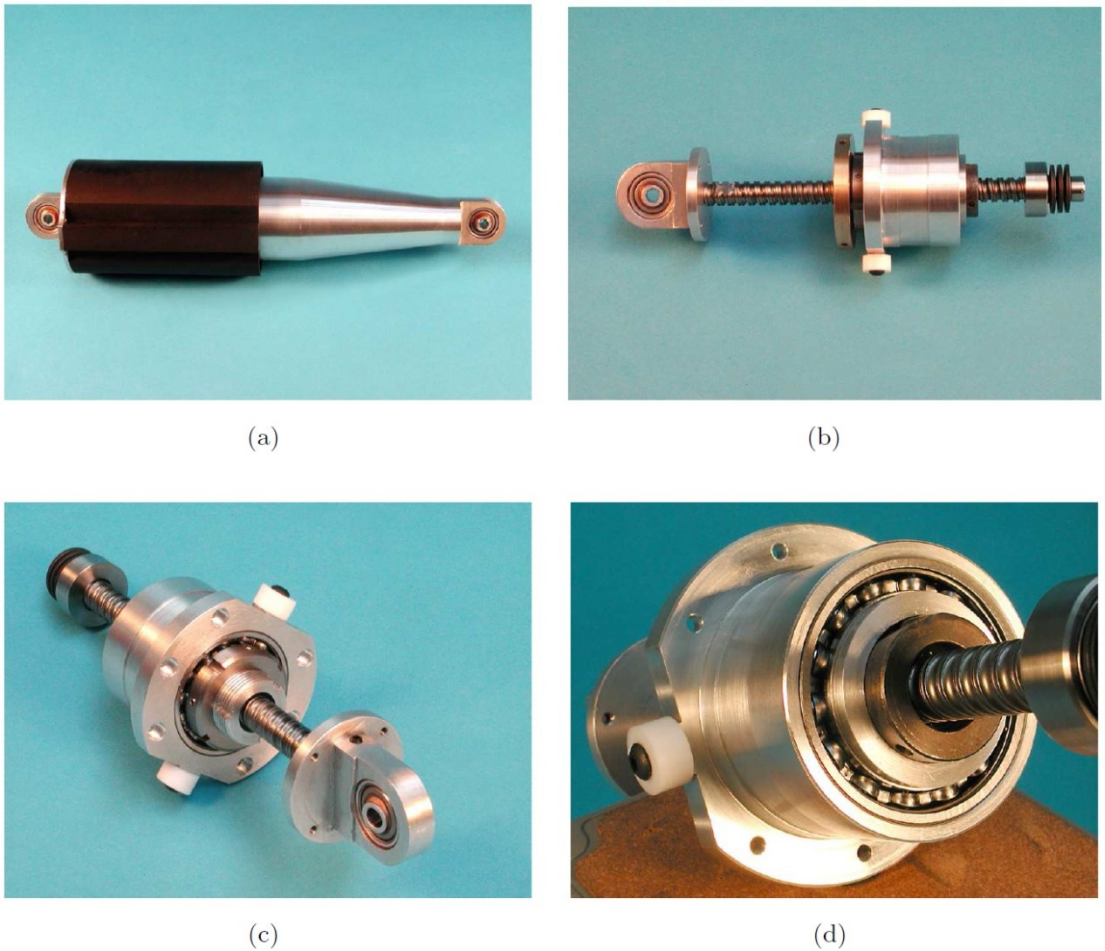


M.C. Smith. Synthesis of mechanical networks : the inerter. *IEEE Transactions on Automatic Control*, 47(10):1648–1662, 2002.

* The Chinese translation 惯容 was coined by Dr. Michael Chen and Prof. Yun Zou in 2009.

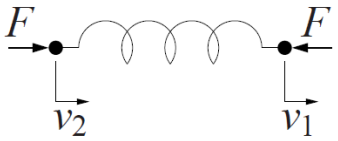
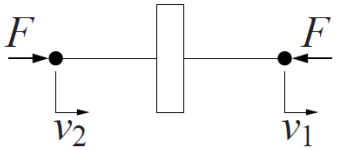
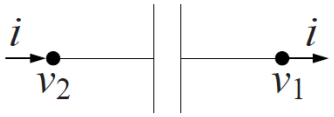
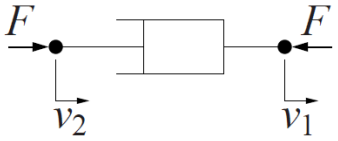


A ball-screw inerter (construction)



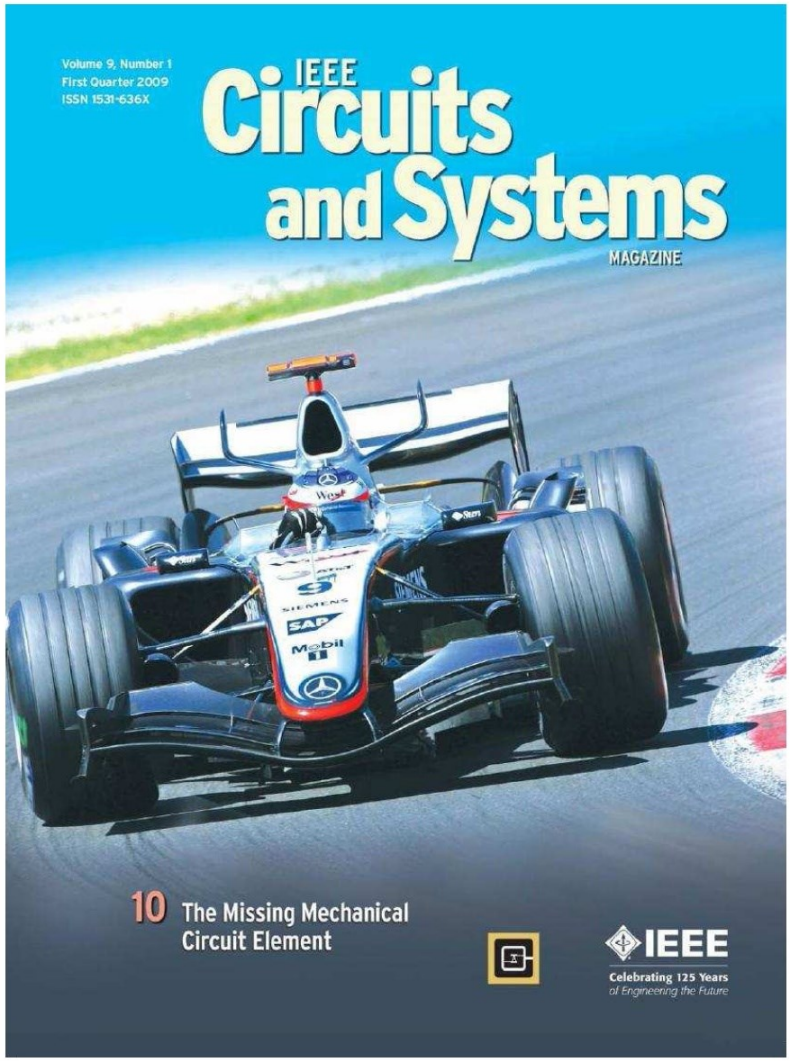
Ball-screw inerter made at Cambridge University Engineering Department; Mass $\approx$ 1 kg, Inertance (adjustable) = 60–240 kg. (a) Complete with outer case, (b) ballscrew, nut and flywheel, (c) flywheel removed, (d) thrust bearing.

A new table of element correspondences

Mechanical		Electrical	
	$Y(s) = \frac{k}{s}$		$Y(s) = \frac{1}{Ls}$
$\frac{dF}{dt} = k(v_2 - v_1)$ spring		$\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor	
	$Y(s) = bs$		$Y(s) = Cs$
$F = b\frac{d(v_2 - v_1)}{dt}$ inerter		$i = C\frac{d(v_2 - v_1)}{dt}$ capacitor	
	$Y(s) = c$		$Y(s) = \frac{1}{R}$
$F = c(v_2 - v_1)$ damper		$i = \frac{1}{R}(v_2 - v_1)$ resistor	

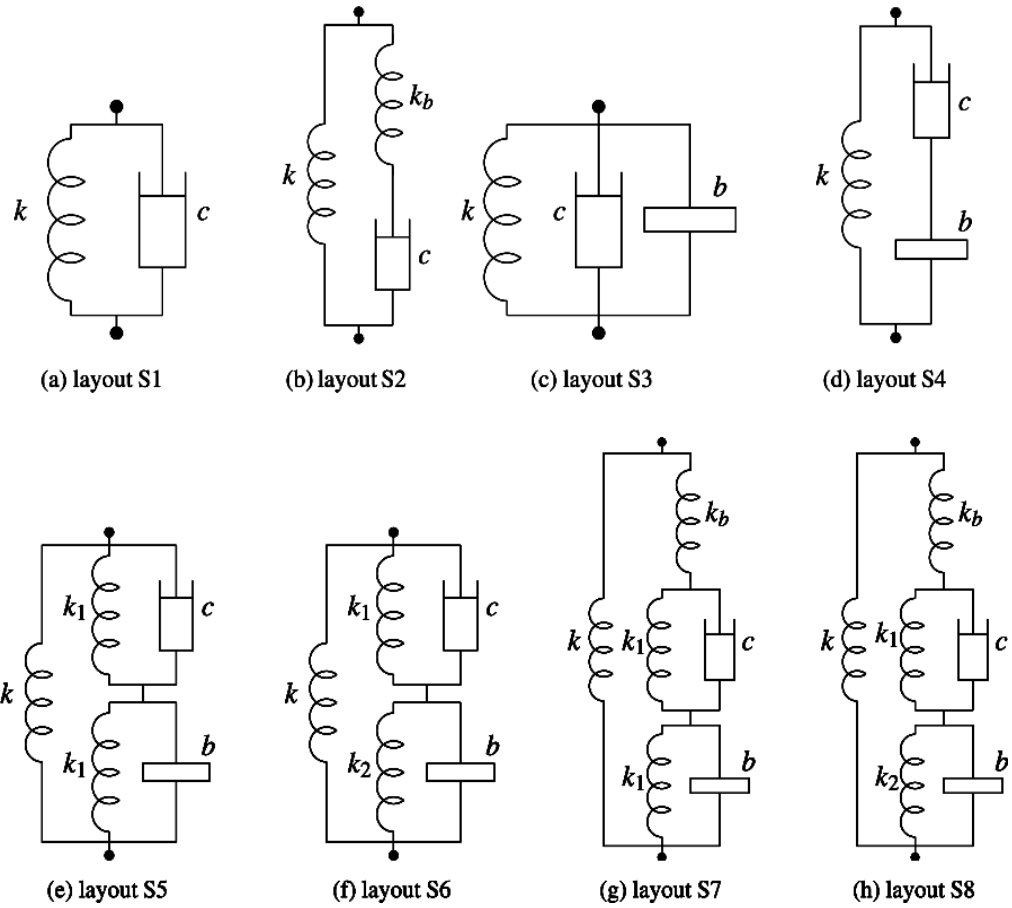
Electrical and mechanical circuit symbols and correspondences. In the force-current analogy forces substitute for currents and velocities substitute for voltages. The admittance $Y(s)$ maps velocity and voltage into force and current, respectively. (The symbol s is the standard Laplace transform variable.)

Want to know more?



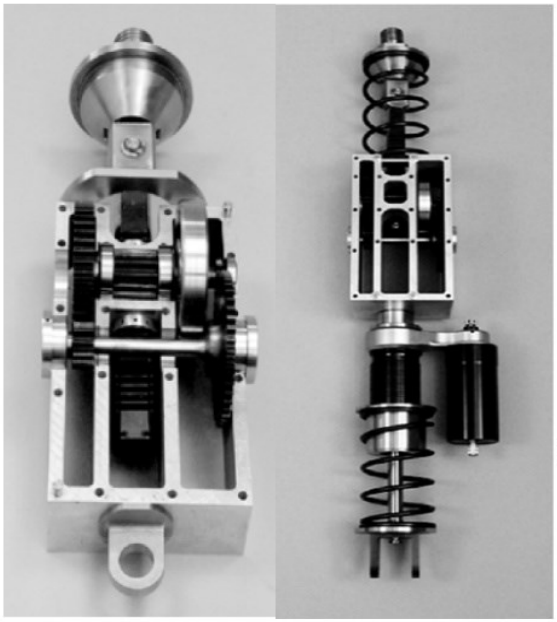
M.Z.Q. Chen, C. Papageorgiou, F. Scheibe, F.-C. Wang, and M.C. Smith, "**The missing mechanical circuit element**," *IEEE Circuits and Systems Magazine*, vol. 9, no. 1, pp. 10–26, 2009. (Cover Feature.)

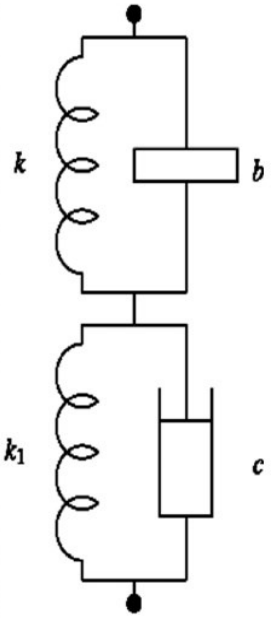
Fixed structure optimization



Layout	Optimal J_1 (ride comfort)
Conventional (layout S1)	2.7358
Parallel inerter (layout S3)	2.5122 (8.17% improvement)
Series inerter centring springs (layout S5)	2.4823 (9.26% improvement)

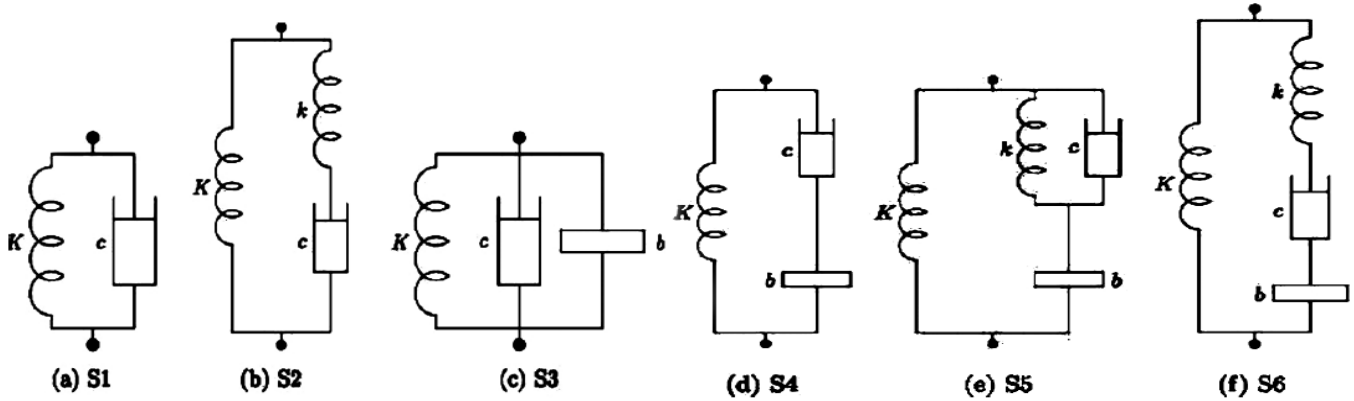
Layout	Optimal J_3 (road holding)
Conventional (layout S1)	1.6288
Parallel inerter (layout S3)	1.6288 (0% improvement)
Series inerter centring springs (layout S5)	1.5224 (6.53% improvement)





M.C. Smith and F.-C. Wang, "Performance benefits in passive vehicle suspensions employing inerters," *Vehicle System Dynamics,* vol. 42, no. 4, pp. 235-257, 2004.

Analytical solutions



An analytical expression of a H_2-norm of the (stable) transfer function $G(s)$ can be computed from a minimal state-space realization $G(s) = C(sI - A)^{-1}B$ as

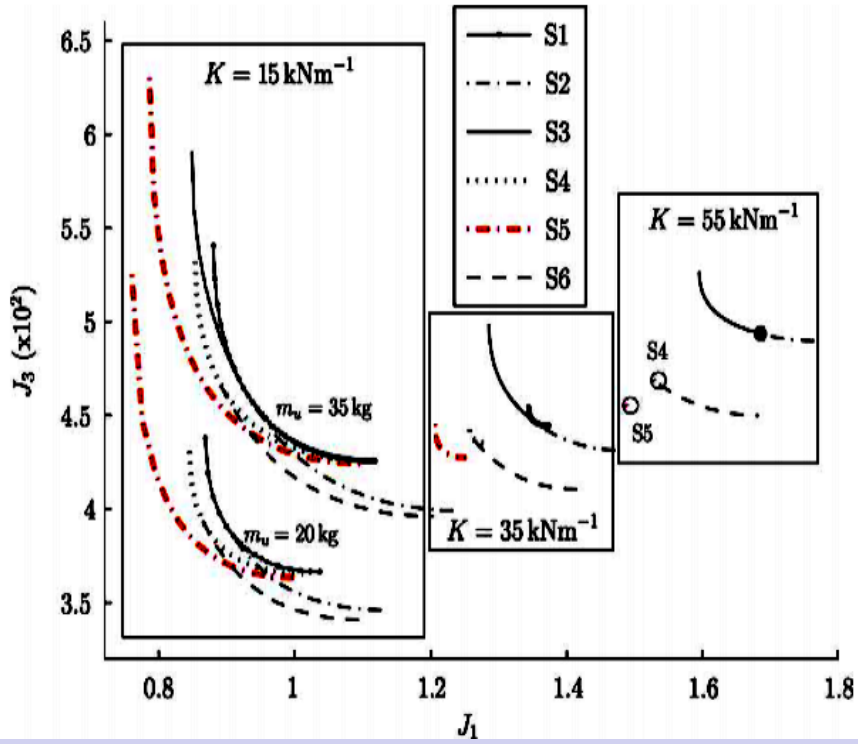
$$\|G\|_2 = \left\|C(sI - A)^{-1}B\right\|_2 = (CLC^T)^{1/2},$$

where the matrix L is the unique solution of the Lyapunov equation

$$AL + LA^T + BB^T = 0.$$

Since $J_i = 2\pi(V\kappa)^{1/2}\,\|T(j\omega)\|_2$, then

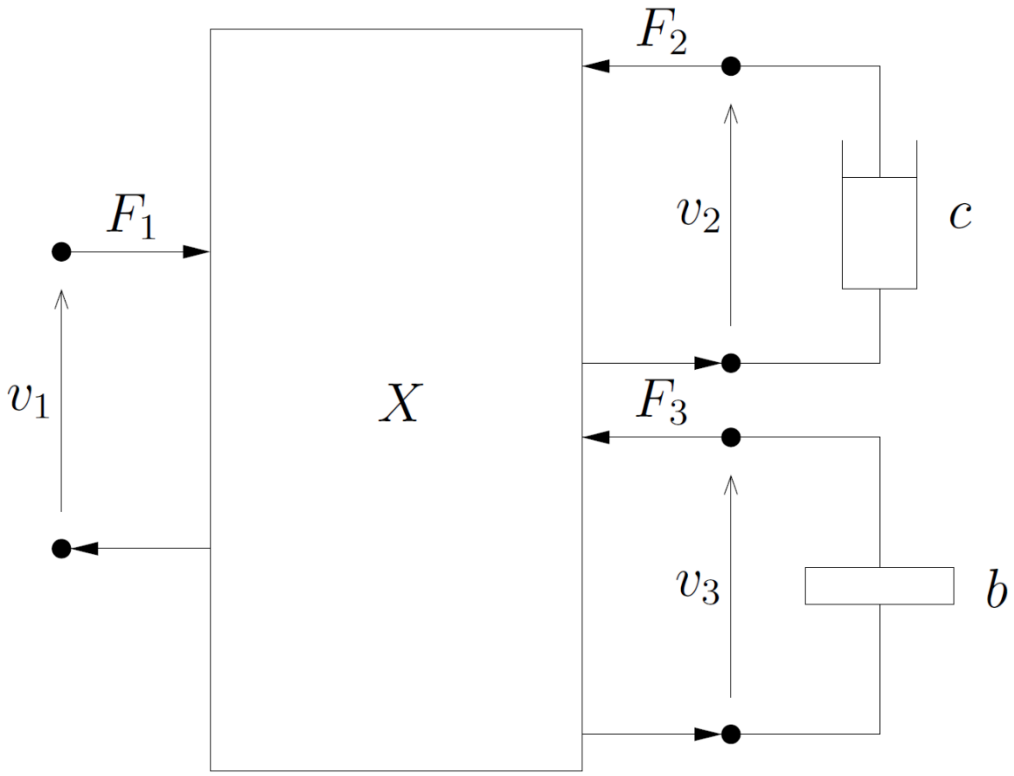
$$J_i = 2\pi\sqrt{V\kappa H_i}, \quad H_i = CLC^T.$$



F. Scheibe and M.C. Smith, "Analytical solutions for optimal ride comfort and tyre grip for passive vehicle suspensions," *Vehicle System Dynamics,* vol. 47, no. 10, pp. 1-24, 2009.

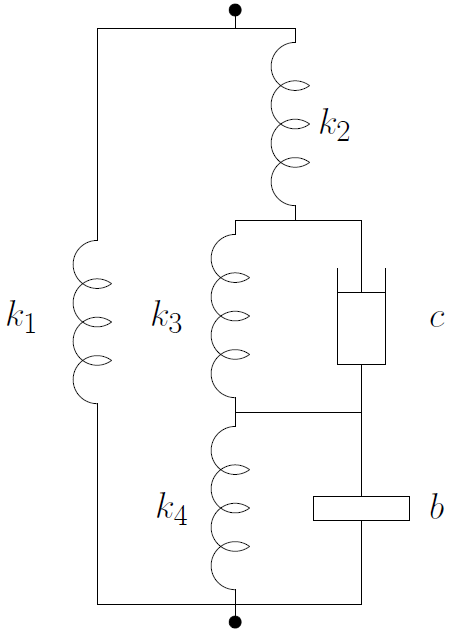
Restricted Complexity Synthesis

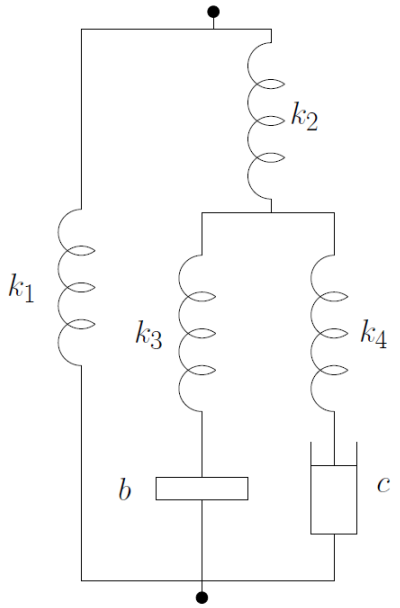
By extracting the damper and inerter, realizability with one damper, one inerter, and a finite number of springs can be converted into realizations of three-port one-element-kind networks:

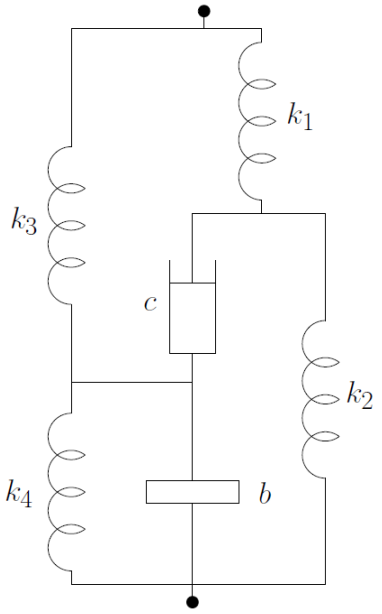


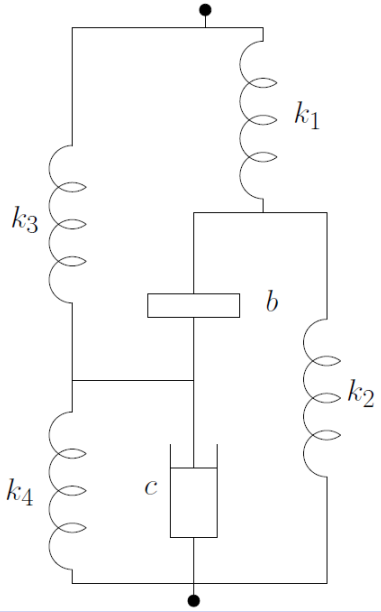
M.Z.Q. Chen and M.C. Smith, "Restricted complexity network realizations for passive mechanical control," *IEEE Trans. on Automatic Control*, vol. 54, no. 10, pp. 2290-2301, 2009. (Regular Paper)

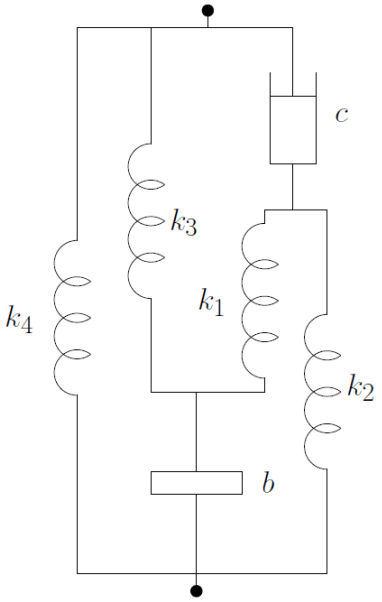
More efficient realisations





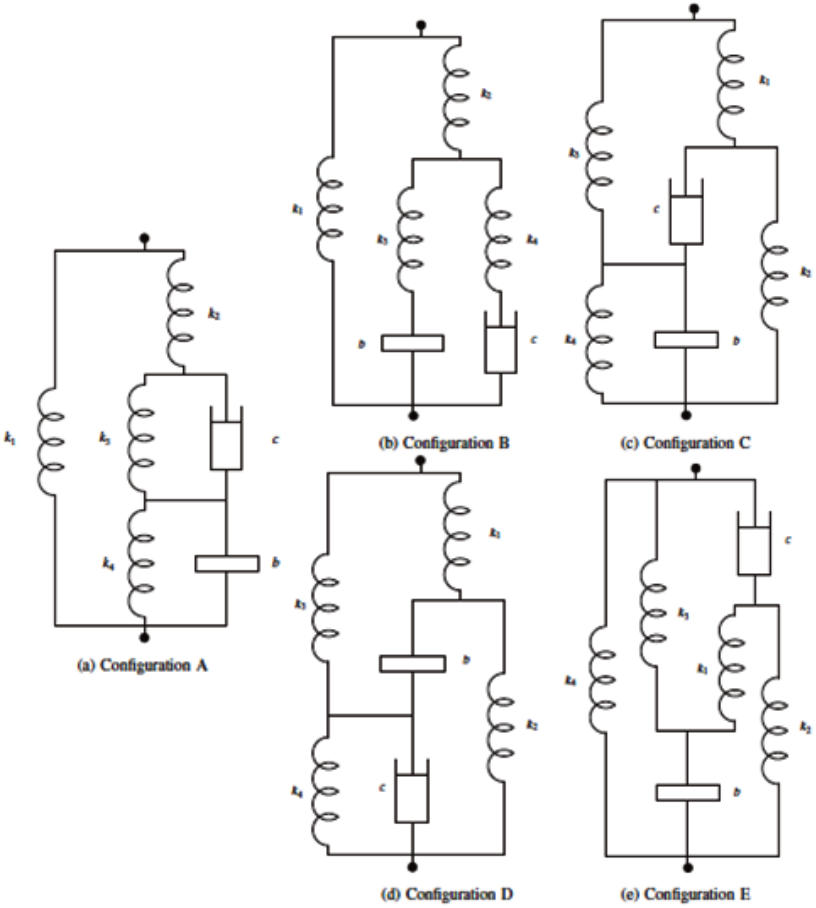






M.Z.Q. Chen and M.C. Smith, "Restricted complexity network realizations for passive mechanical control," *IEEE Trans. on Automatic Control*, vol. 54, no. 10, pp. 2290-2301, 2009. (Regular Paper)

Suspensions with one damper and one inerter



Ride comfort: r.m.s. body vertical acceleration

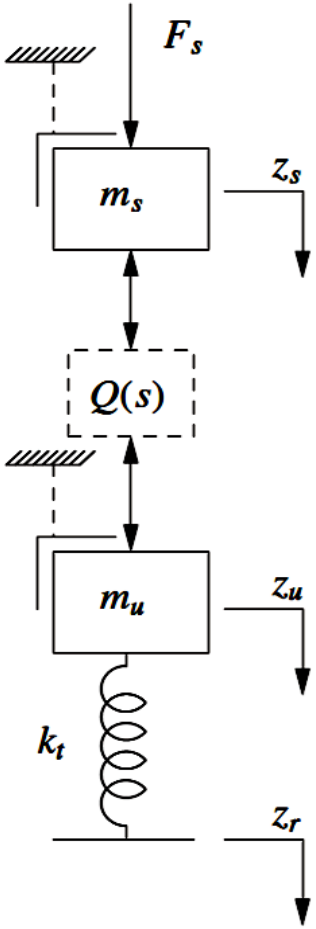
$$J_1 = 2\pi(V\kappa)^{1/2}\| T_{\dot{z}_r \to \ddot{z}_s} \|_2$$

Road holding: r.m.s. dynamic tyre load

$$J_3 = 2\pi(V\kappa)^{1/2}\| T_{\dot{z}_r \to k_t(z_u - z_r)} \|_2$$

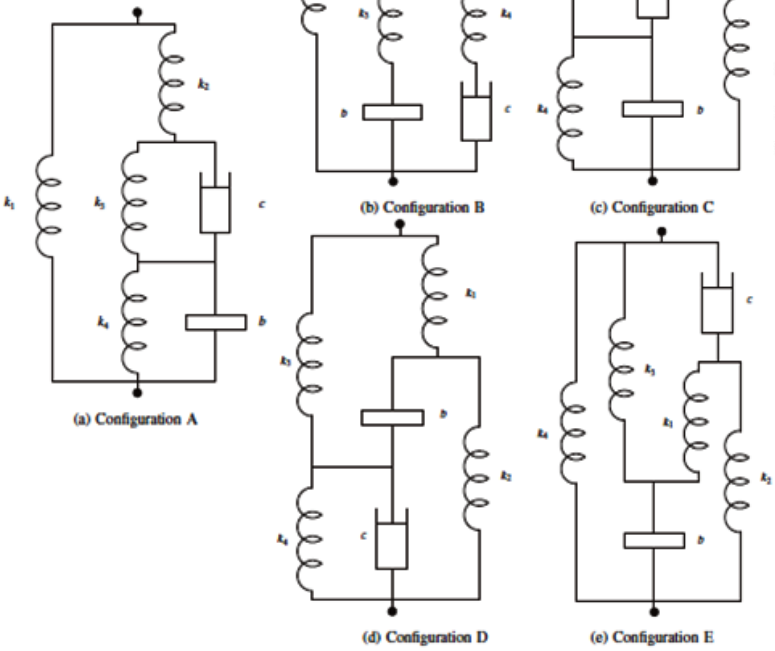
Handling: Dynamic load carrying

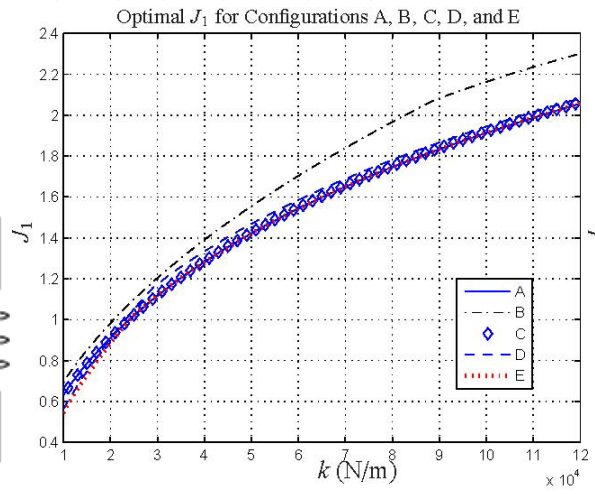
$$J_5 = \| T_{F_s \to z_s} \|_\infty$$

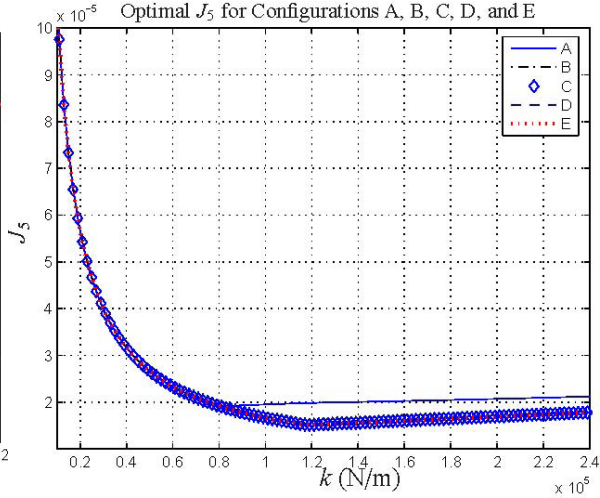


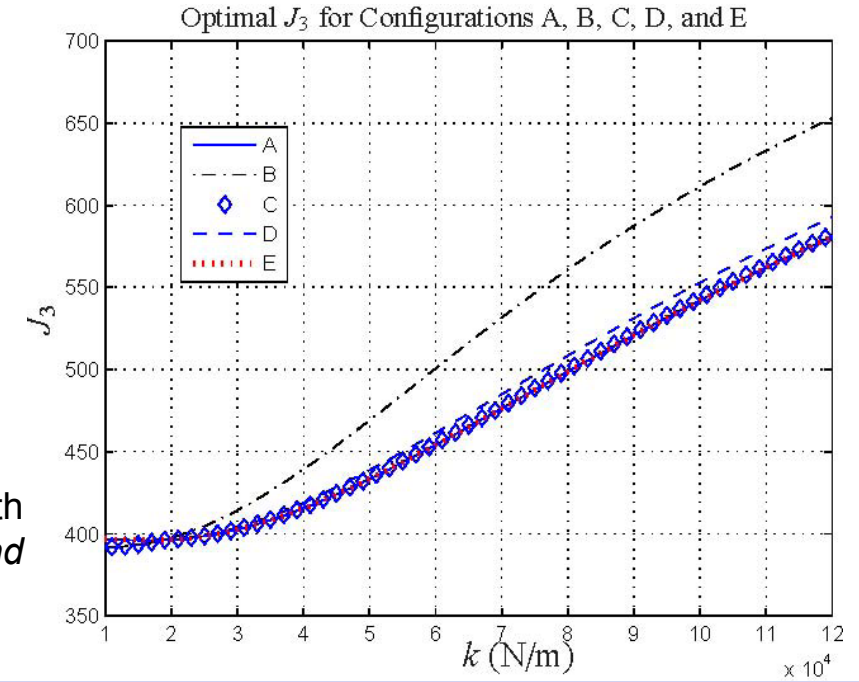
M.Z.Q. Chen, Y. Hu, and B. Du, "Suspension performance with one damper and one inerter," *the 24th Chinese Control and Decision Conference*, Taiyuan, China, 2012, pp. 3551-3556.

Suspensions with one damper and one inerter



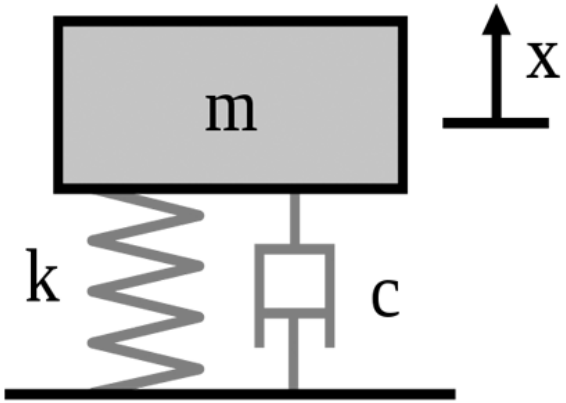


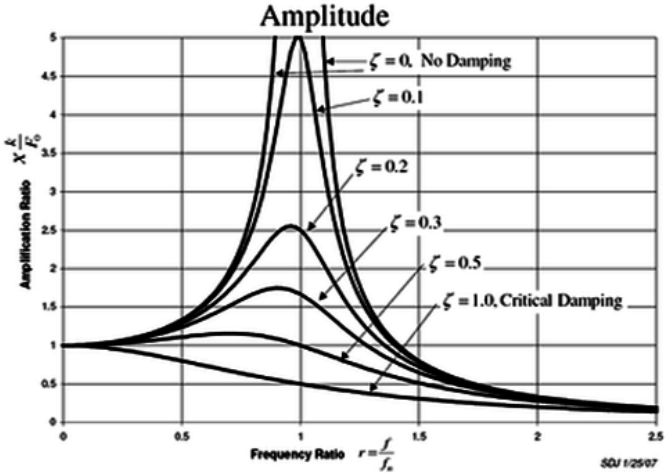




M.Z.Q. Chen, Y. Hu, and B. Du, "Suspension performance with one damper and one inerter," *the 24th Chinese Control and Decision Conference*, Taiyuan, China, 2012, pp. 3551-3556.

Function of inerter in vibration





Free vibration:

$$m\ddot{x} + c\dot{x} + kx = 0,$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = 0.$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

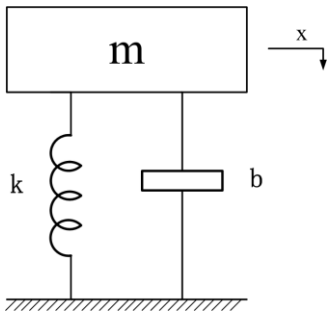
◆Spring: Store energy; Static support; Determine natural frequency.

◆Damper: Dissipate energy; Decrease natural frequency.

◆ Inerter: Store energy; ?

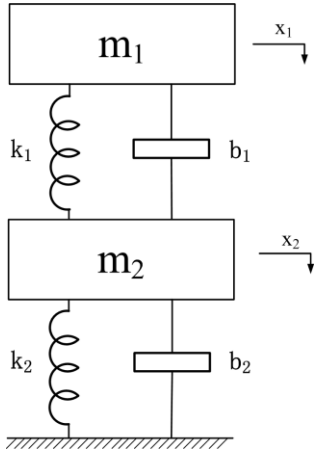
M.Z.Q. Chen, Y. Hu, L. Huang, and G. Chen, "Influence of inerter on natural frequencies of vibration systems," *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

SDOF and TDOF system



The natural frequency ω_n of an SDOF system is a decreasing function of the inertance b. Thus, inerter can reduce the natural frequency of an SDOF system.

$$\omega_n = \sqrt{\frac{k}{m+b}}$$



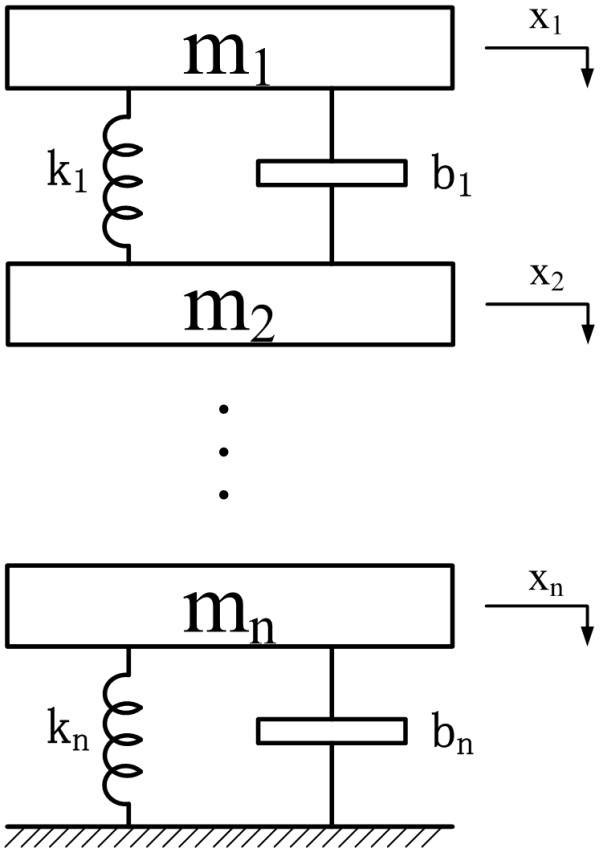
For a TDOF system with two inerters, both natural frequencies ω_{n1} and ω_{n2} are decreasing functions of the inertance b_1 and b_2.

$$\omega_{n1} = \sqrt{\frac{k_1 k_2 (f_1 + f_2 - \sqrt{(f_1 - f_2)^2 + 4d_0}}{2(f_1 f_2 - d_0)}}$$

$$\omega_{n2} = \sqrt{\frac{k_1 k_2 (f_1 + f_2 + \sqrt{(f_1 - f_2)^2 + 4d_0}}{2(f_1 f_2 - d_0)}}$$

M.Z.Q. Chen, Y. Hu, L. Huang, and G. Chen, "Influence of inerter on natural frequencies of vibration systems," *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

MDOF system



Sensitivity analysis on natural frequencies (eigenvalues) and mode shapes (eigenvectors)

Consider the MDOF system. For an arbitrary eigenvalue λ_j, $j = 1, \ldots, n$, and an arbitrary inertance b_i, $i = 1, \ldots, n$, the following equations hold:

$$\frac{\partial \lambda_j}{\partial b_i} = -\lambda_j \Phi_{ij},$$

where Φ_{ij}, $j = 1, \ldots, n$, is defined as

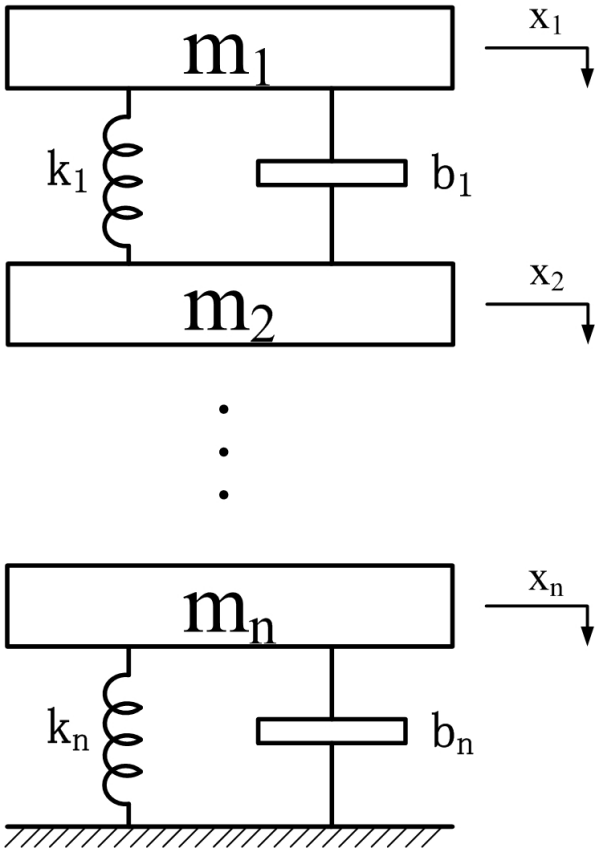
$$\Phi_{ij} = \boldsymbol{\varphi_j}^T \frac{\partial \mathbf{M}}{\partial b_i} \boldsymbol{\varphi_j} = \begin{cases} \left(\varphi_j^{(i)} - \varphi_j^{(i+1)}\right)^2, & i \neq n \\ \left(\varphi_j^{(n)}\right)^2, & i = n \end{cases}$$

1. The natural frequencies of the MDOF system can always be reduced by increasing the inertance of any inerter.

2. The natural frequencies of any MDOF system can be reduced by an inerter if the inertial matrix satisfies $\frac{\partial \mathbf{M}}{\partial b_i} \geq 0$.

M.Z.Q. Chen, Y. Hu, L. Huang, and G. Chen, "Influence of inerter on natural frequencies of vibration systems," *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

MDOF system



Physical interpretation: for a small increment of inertance ε_{b_i} of a particular inerter b_i, one obtains

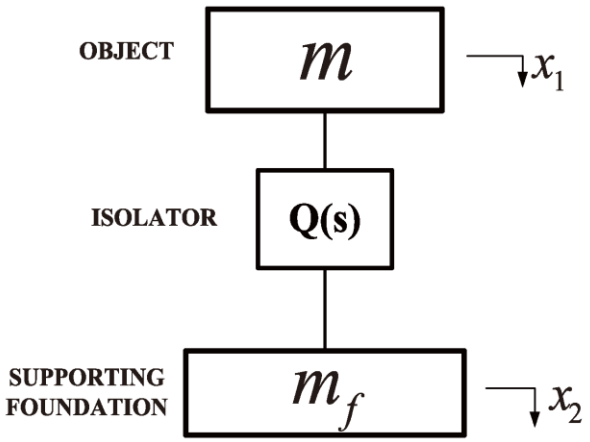
$$\mathbf{M} = \mathbf{M}_0 + \varepsilon_{b_i} \frac{\partial \mathbf{M}}{\partial b_i},$$

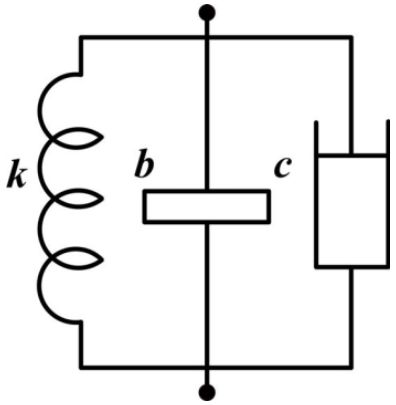
Sine $\frac{\partial \mathbf{M}}{\partial b_i}$ is positive semidefine, it can be interpreted as increasing the mass of the whole system.

$$\mathbf{M} = \begin{bmatrix} m_1 + b_1 & -b_1 & & \\ -b_1 & m_2 + b_1 + b_2 & -b_2 & \\ & \ddots & \ddots & \ddots \\ & & -b_{n-1} & m_n + b_{n-1} + b_n \end{bmatrix}$$

M.Z.Q. Chen, Y. Hu, L. Huang, and G. Chen, “Influence of inerter on natural frequencies of vibration systems,” *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

Vibration analysis





$$Q_1(s) = \frac{k}{s} + bs + c$$

The displacement transmissibility

$$\mu_x = \frac{|\, x_1 \,|}{|\, x_2 \,|} = \frac{|\, Q(j\omega)j\omega \,|}{|\, Q(j\omega)j\omega - m\omega^2 \,|},$$

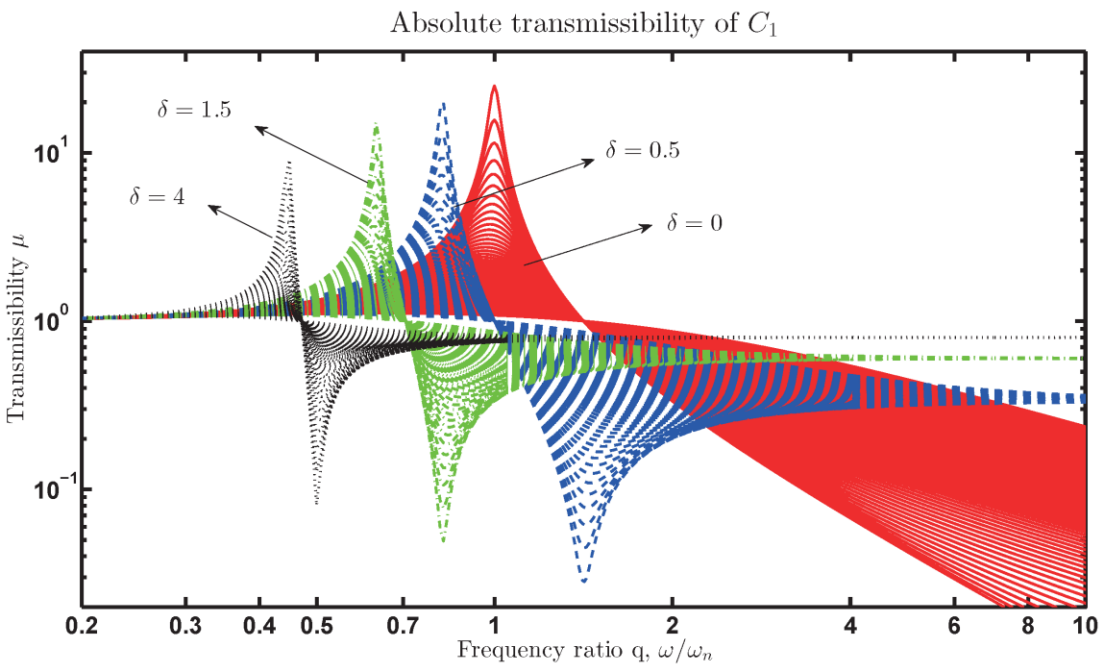
The force transmissibility

$$\mu_F = \frac{|\, F_f \,|}{|\, F_0 \,|} = \frac{m_f}{m + m_f} \frac{|\, Q(j\omega)j\omega \,|}{|\, Q(j\omega)j\omega - \frac{mm_f}{m+m_f}\omega^2 \,|}.$$

$$\mu = \sqrt{\frac{(1 - \delta q^2)^2 + (2\zeta q)^2}{(1 - (1+\delta)q^2)^2 + (2\zeta q)^2}}$$

$$q = \frac{\omega}{\omega_n},\ \omega_n = \sqrt{\frac{k}{m}},\ c_r = 2\omega_n m = 2\sqrt{mk},\ \zeta = \frac{c}{c_r},\ \delta = \frac{b}{m}.$$

Y. Hu, M.Z.Q. Chen, Z. Shu, and L. Huang, "Vibration analysis for isolation system with inerter," *Proceedings of the 33rd Chinese Control Conference*, Nanjing, China, 2014, pp. 6687-6692.



Absolute transmissibility μ where ζ ranges from 0.02 to 1.2.

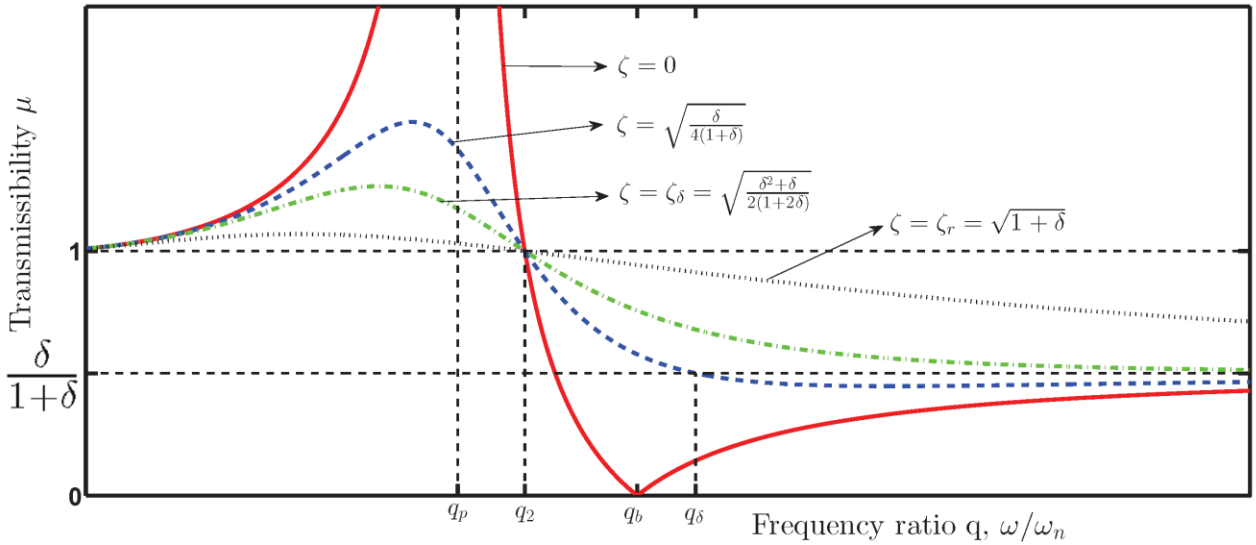
$$q = \frac{\omega}{\omega_n}, c_r = 2\sqrt{mk}, \zeta = \frac{c}{c_r}, \delta = \frac{b}{m}$$

$$q_p = \frac{\omega_{np}}{\omega_n} = \sqrt{\frac{1}{1+\delta}}$$

$$q_2 = \sqrt{\frac{2}{1+2\delta}}$$

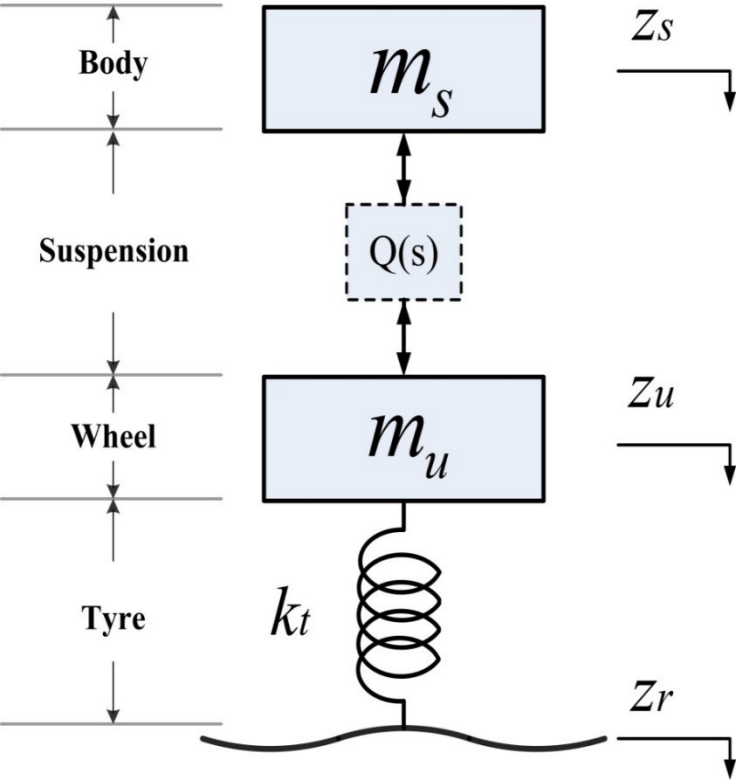
$$q_b = \frac{\omega_b}{\omega_n} = \sqrt{\frac{1}{\delta}}$$

$$q_\delta = \frac{\sqrt{2}}{2}\sqrt{\frac{1+2\delta}{\delta^2+\delta-2\zeta^2(1+2\delta)}}$$



Y. Hu, M.Z.Q. Chen, Z. Shu, and L. Huang, "Vibration analysis for isolation system with inerter," *Proceedings of the 33rd Chinese Control Conference*, Nanjing, China, 2014, pp. 6687-6692.

Multiple requirements design



Ride comfort: r.m.s. body vertical acceleration

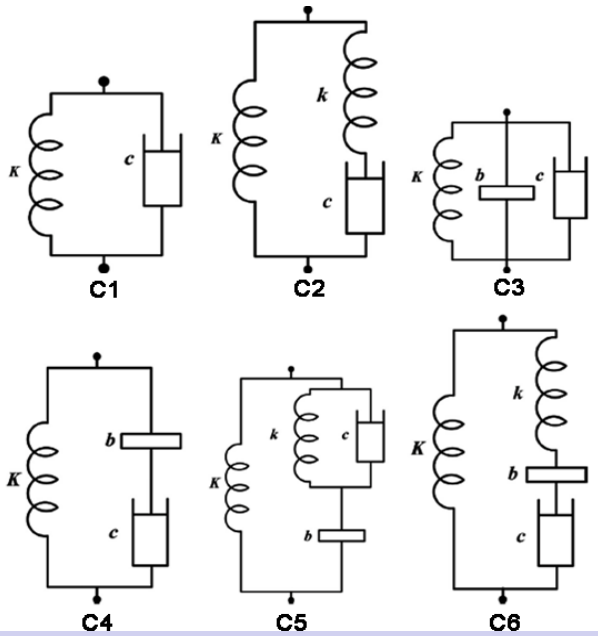
$$J_1 = 2\pi(V\kappa)^{1/2}||T_{\dot{z}_r\to\ddot{z}_s}||_2 = 2\pi(V\kappa)^{1/2}\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|T_{\dot{z}_r\to\ddot{z}_s}(j\omega)|^2\,d\omega}$$

Suspension deflection: r.m.s. suspension deflection

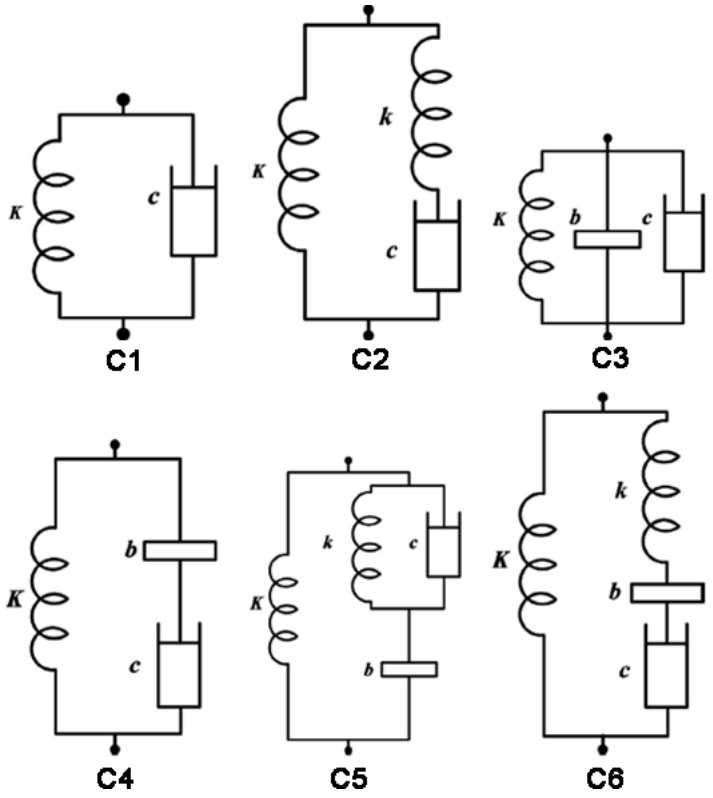
$$J_2 = 2\pi(V\kappa)^{1/2}||T_{\dot{z}_r\to(z_s-z_u)}||_2 = 2\pi(V\kappa)^{1/2}\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|T_{\dot{z}_r\to(z_s-z_u)}(j\omega)\right|^2 d\omega}$$

Road holding: r.m.s. dynamic tyre load

$$J_3 = 2\pi(V\kappa)^{1/2}||T_{\dot{z}_r\to k_t(z_u-z_r)}||_2 = 2\pi(V\kappa)^{1/2}\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|T_{\dot{z}_r\to k_t(z_u-z_r)}(j\omega)\right|^2 d\omega}$$



Y. Hu, M.Z.Q. Chen, and Z. Shu, "Passive vehicle suspensions employing inerters with multiple performance requirements," *Journal of Sound and Vibration,* vol. 333, no. 8, pp. 2212–2225, 2014.



Suspension deflection

1. For the individual performance J_2, the optimal values of each configuration are all zero by carefully choosing the coefficients of elements.
2. $C2$, $C4$ and $C6$ provide no improvement for suspension deflection compared with $C1$, and $C3$ achieves equal suspension deflection with $C1$.
3. $C5$ performs better than $C4$ if and only if $k < 2e_3(e_4 b^{-1} + e_5)^{-1}$.
4. $C6$ performs better than $C4$ if and only if $k > e_2 b(2e_1)^{-1}$.

Ride comfort

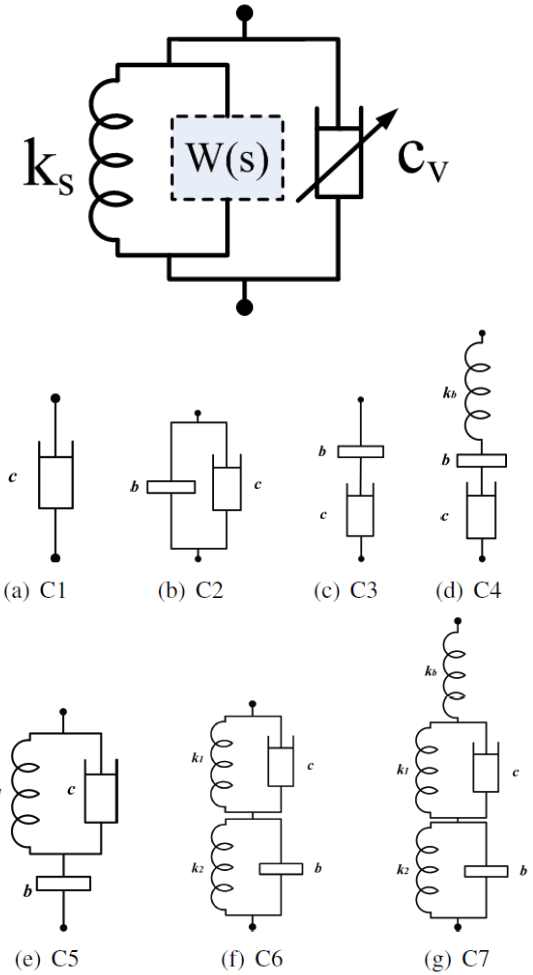
1. $C2$ provides no improvement for the ride comfort compared with $C1$.
2. $C3$ performs better than $C1$ if and only if $b < b_1 < \sqrt{4Km_s m_u k_t^{-1}}$.
3. $C4$ performs better than $C1$ if and only if $b > -d_6/d_5 = 1/2(K/k_t + 1/(1+m_u/m_s)^2)(m_s+m_u) > 1/2(m_s+m_u)$.
4. $C5$ performs better than $C4$ if and only if $b < -2d_6/d_5$ and $k < (d_5 + 2d_6 b^{-1})(d_7 + d_8 b^{-1} + d_9 b^{-2})^{-1}$.
5. $C6$ performs better than $C2$ if and only if $b > d_6(2d_2 k^{-1} - d_5)^{-1}$; $C6$ performs better than $C4$ if and only if $b < K(m_s+m_u)k_t^{-1}$ and $k > d_4(2d_2 b^{-1} - d_3)^{-1}$.

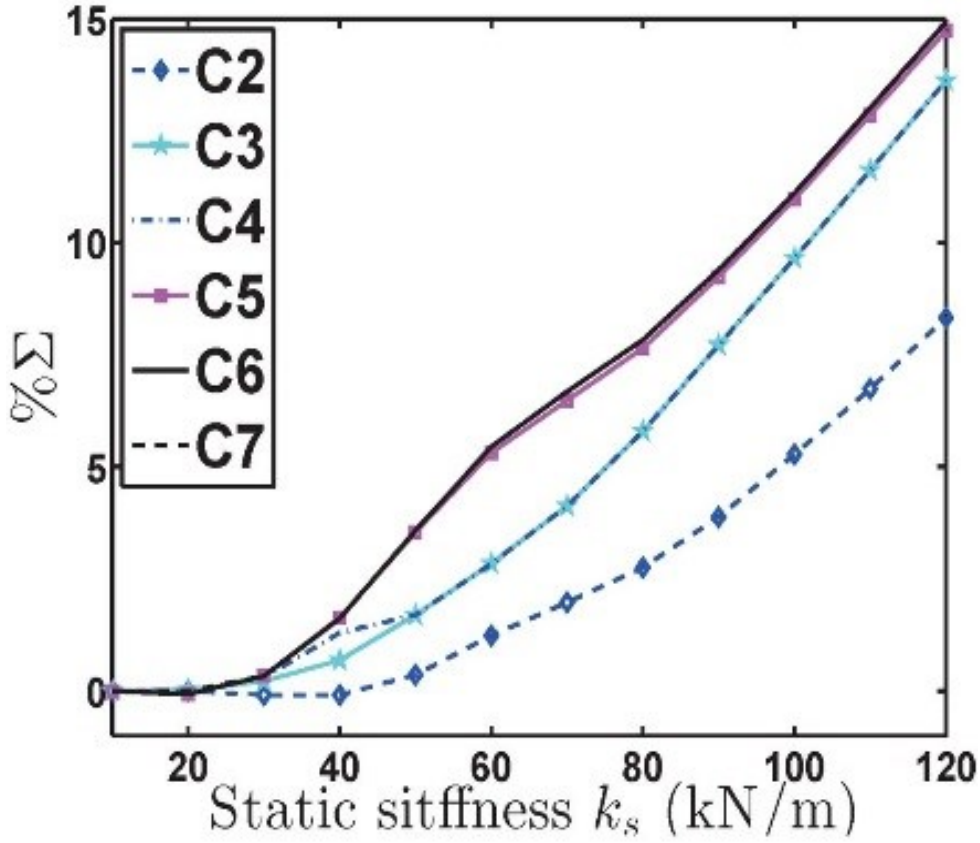
Road holding

1. $C2$ performs better than $C1$ if and only if $K < m_s(m_s+2m_u)k_t/(2(m_s+m_u)^2)$ and $k > -a_4/a_3$.
2. $C3$ performs better than $C1$ if and only if $K > m_s(m_s+2m_u)k_t/(2(m_s+m_u)^2)$ and $b < a_3/a_5$.
3. $C4$ performs better than $C1$ if and only if $K > m_s m_u k_t/(m_s+m_u)^2$ and $b > -a_7/a_6$.
4. $C5$ performs better than $C4$ if and only if $k < (a_6 + 2a_7 b^{-1})(a_8 + a_9 b^{-1} + a_{10} b^{-2})^{-1}$ and $b < -2a_7/a_6$.
5. $C6$ performs better than $C2$ if and only if $b > a_7(2a_2 k^{-1} - a_6)^{-1}$;$C6$ performs better than $C4$ if and only if $k > a_4(2a_2 b^{-1} - a_3)^{-1}$.

Y. Hu, M.Z.Q. Chen, and Z. Shu, "Passive vehicle suspensions employing inerters with multiple performance requirements," *Journal of Sound and Vibration*, vol. 333, no. 8, pp. 2212–2225, 2014.

Semi-active suspension with inerter





M.Z.Q. Chen, Y. Hu, C. Li, and G. Chen, "Performance benefits of using inerter in semi-active suspensions," *IEEE Trans. Control Syst. Technol.*, in press. DOI:10.1109/TCST.2014.2364954.

Semi-active suspension with semi-active inerter

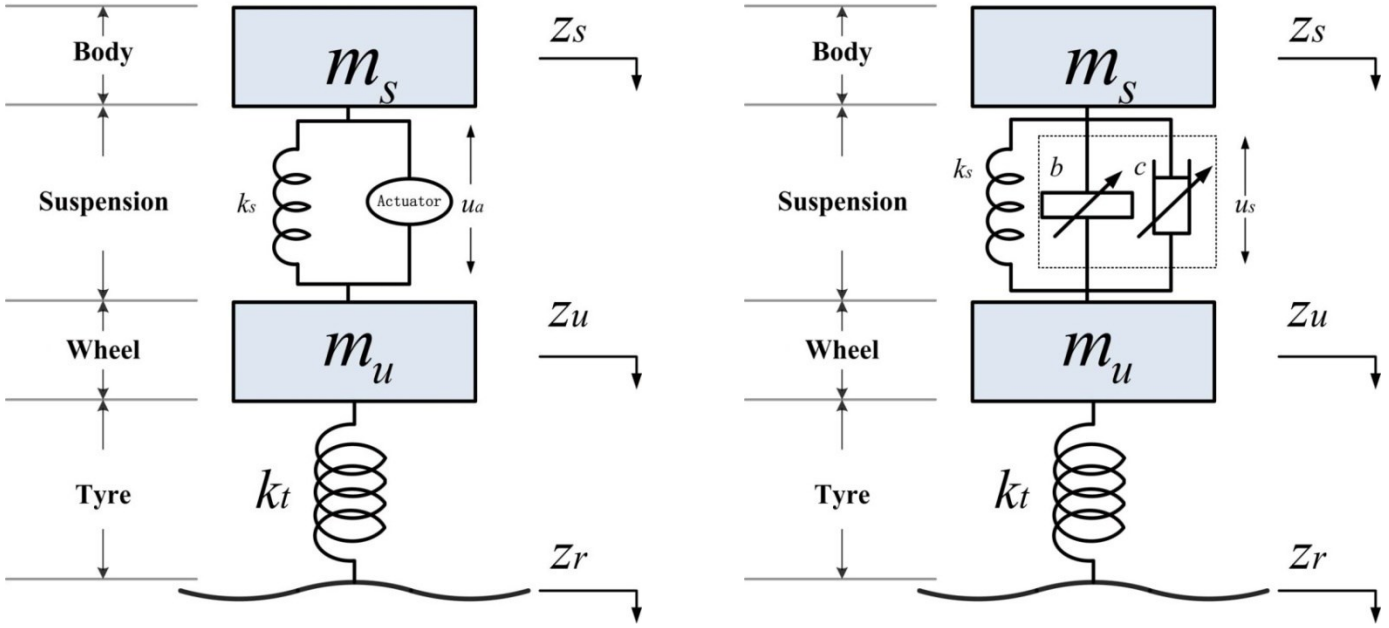


Table 1. Semi-active suspension schemes.

Scheme	Semi-active damper	Semi-active inerter
Semi-I & D	Employed	Employed
Semi-I	Not employed	Employed
Semi-D	Employed	Not employed

M. Z. Q. Chen, Y. Hu, C. Li, and G. Chen, "Semi-active suspension with semi-active inerter and semi-active damper," *the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 11225–11230.

Semi-active suspension with semi-active inerter

$$\dot{z}_r(t) = -\alpha V z_r(t) + w(t)$$

Road classified by ISO8608.

Road Class	σ (10^{-3} m)	$\Phi(\Omega_0)$ (10^{-6} m^3), $\Omega_0 = 1$	α (rad/m)
A (very good)	2	1	0.127
B (good)	4	4	0.127
C (average)	8	16	0.127
D (poor)	16	64	0.127
E (very poor)	32	256	0.127

Road	Performance	Active	Semi-I & D	Semi-D	Semi-I	Passive
B road	$RM.J$ ($\times 10^{-1}$)	6.55	6.56	6.59	7.52	7.37
	J_{acc} (m/s^2)	0.41	0.41	0.42	0.36	0.57
	J_{sws} (mm)	3.11	3.12	3.16	4.28	2.69
	J_{rhd} (mm)	1.58	1.57	1.57	2.05	1.44
	u_e (N)	0	8.18	17.16	77.24	65.64
D road	$RM.J$	2.62	2.62	2.64	3.01	2.95
	J_{acc} (m/s^2)	1.65	1.65	1.69	1.43	2.29
	J_{sws} (cm)	1.24	1.25	1.26	1.71	1.07
	J_{rhd} (mm)	6.32	6.31	6.27	8.19	5.78
	u_e (N)	0	32.71	68.62	308.96	262.56

M. Z. Q. Chen, Y. Hu, C. Li, and G. Chen, "Semi-active suspension with semi-active inerter and semi-active damper," *the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 11225–11230.

Representative applications

Vehicle suspension:

Smith & Wang 2004, Papageorgiou & Smith 2006, Wang & Su 2008, Wang & Chan 2011, Chen *et al.* 2012, Wang *et al.* 2014, Hu *et al.* 2014

Train suspension:

Wang *et al.* 2009, Wang & Liao 2010, Jiang *et al*. 2012, Wang *et a*l. 2012

Motorcycle steering compensator:

Evangelou *et al.* 2004, Evangelou *et al.* 2006, Evangelou *et al*. 2007

Building vibration control:

Wang *et al.* 2010, Takewaki *et al.* 2012, Sugimura *et al.* 2012, Lazar *et al.* 2013

Vibratory energy harvesting:

Cassidy *et al.* 2011, Jang *et al.* 2011

Long-span suspension bridge:

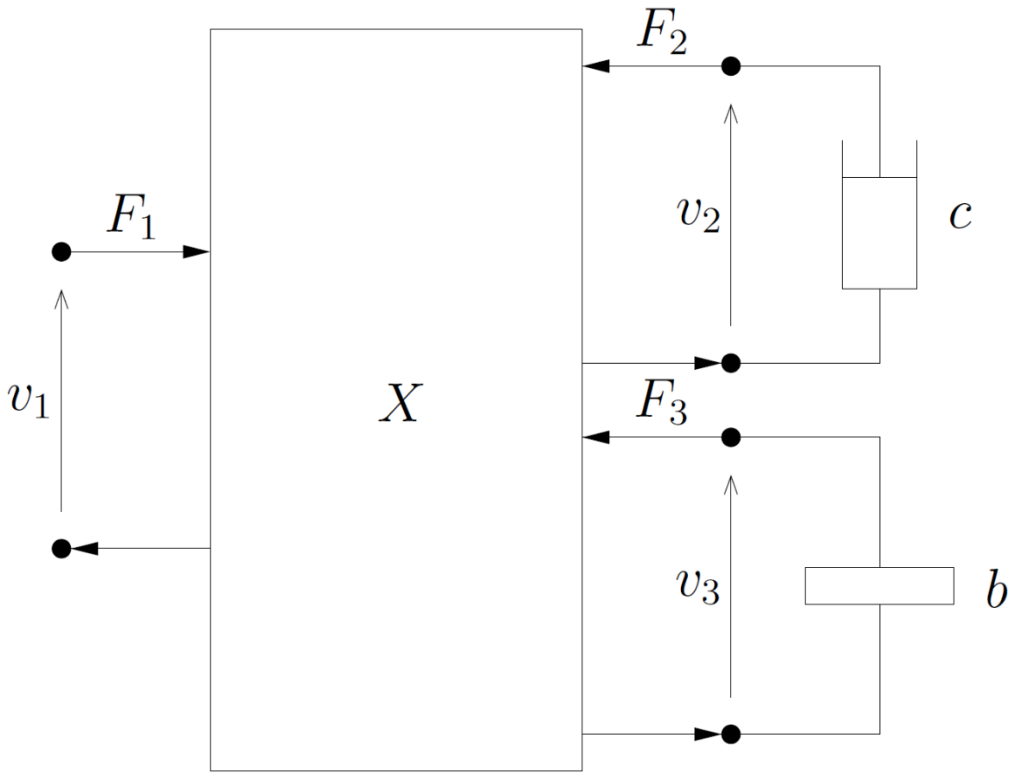
Zhao *et al.* 2011, Limebeer *et al.* 2011

Passive dynamic walker:

Hanazawa *et al.* 2011

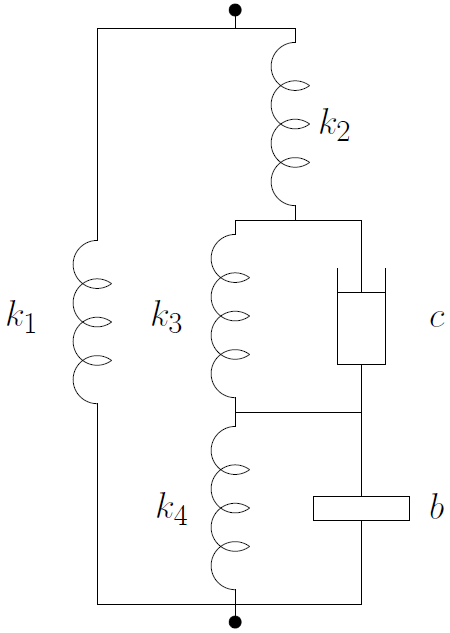
Element Extraction Technique

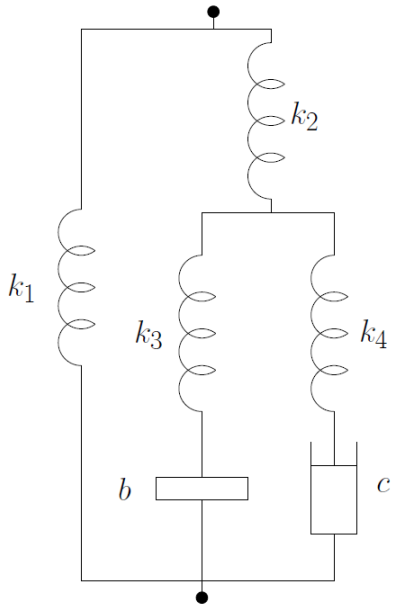
By extracting the damper and inerter, realizability with one damper, one inerter, and a finite number of springs can be converted into realizations of three-port one-element-kind networks:

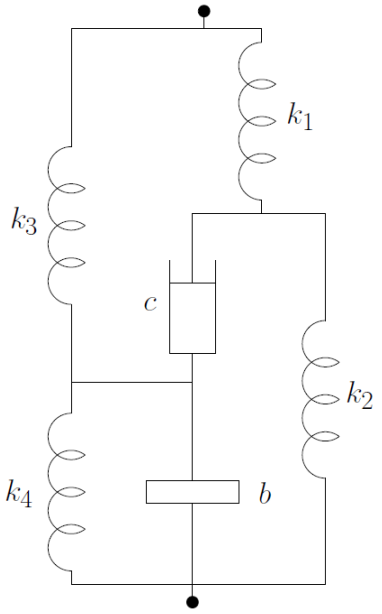


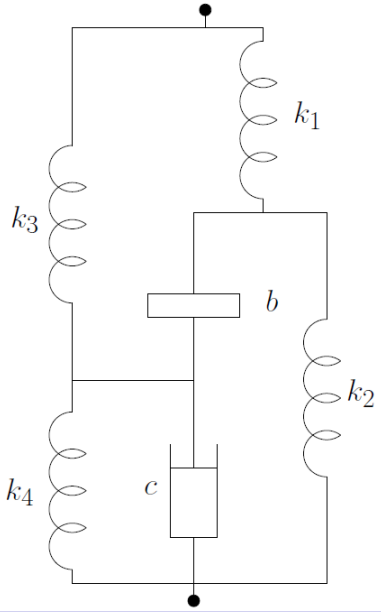
M.Z.Q. Chen and M.C. Smith, "Restricted complexity network realizations for passive mechanical control," *IEEE Trans. on Automatic Control*, vol. 54, no. 10, pp. 2290-2301, 2009. (Regular Paper)

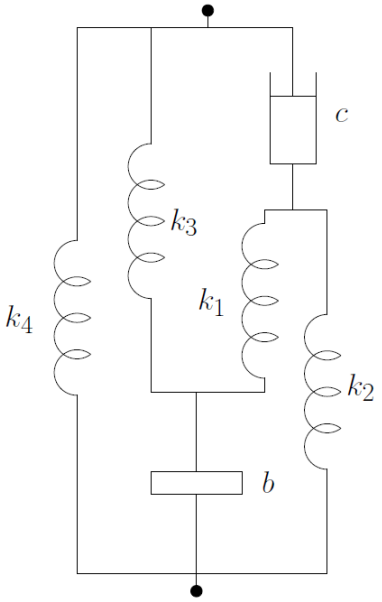
More efficient realisations







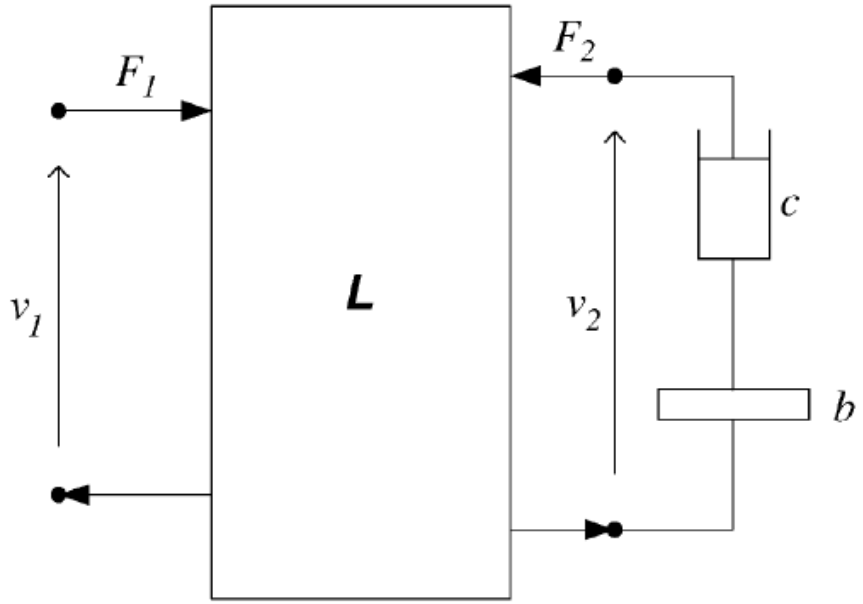




M.Z.Q. Chen and M.C. Smith, "Restricted complexity network realizations for passive mechanical control," *IEEE Trans. on Automatic Control*, vol. 54, no. 10, pp. 2290-2301, 2009. (Regular Paper)

Element Extraction Technique (Cont.)

For the special case when the impedance of three-port networks does not exist, the structure is as below.

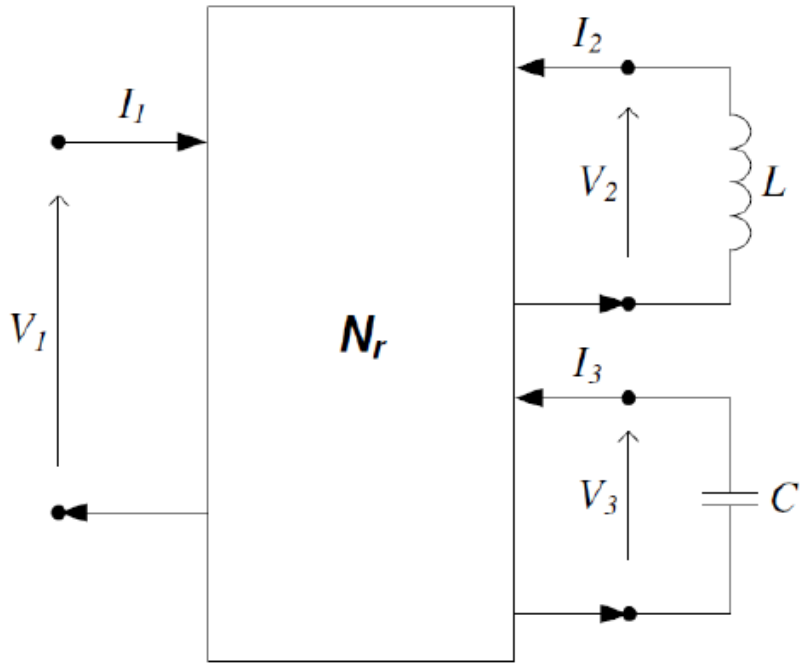


This is converted into realizations of two-port one-element-kind networks.

M.Z.Q. Chen, K. Wang, Y. Zou, and J. Lam. “Realization of a special class of admittances with one damper and one inerter for mechanical control," *IEEE Trans. Automatic Control*, vol. 58, no. 7, pp. 1841–1846, 2013.

Element Extraction Technique (Cont.)

For electrical networks, the minimal reactance synthesis can be solved by extracting the reactive elements.



M. Z. Q. Chen and M. C. Smith, "Electrical and mechanical passive network synthesis," in *Recent Advances in Learning and Control*, V. D. Blondel, S. P. Boyd, and H. Kimura (Eds.), New York: Springer-Verlag, 2008, LNCIS, vol. 371, pp. 35–50.

Motivation of Investigating *n*-Port Resistive Networks

1. Based on the element extraction technique, the results can be utilized to solve minimal realization problems of passive mechanical (electrical) networks.

2. The investigation is the first step towards solving *n*-port passive networks without levers (transformers).

Paramountcy

Definition

A real symmetric matrix is paramount if each of its principal minors is not less than than the absolute value of any other minor built from the same rows.

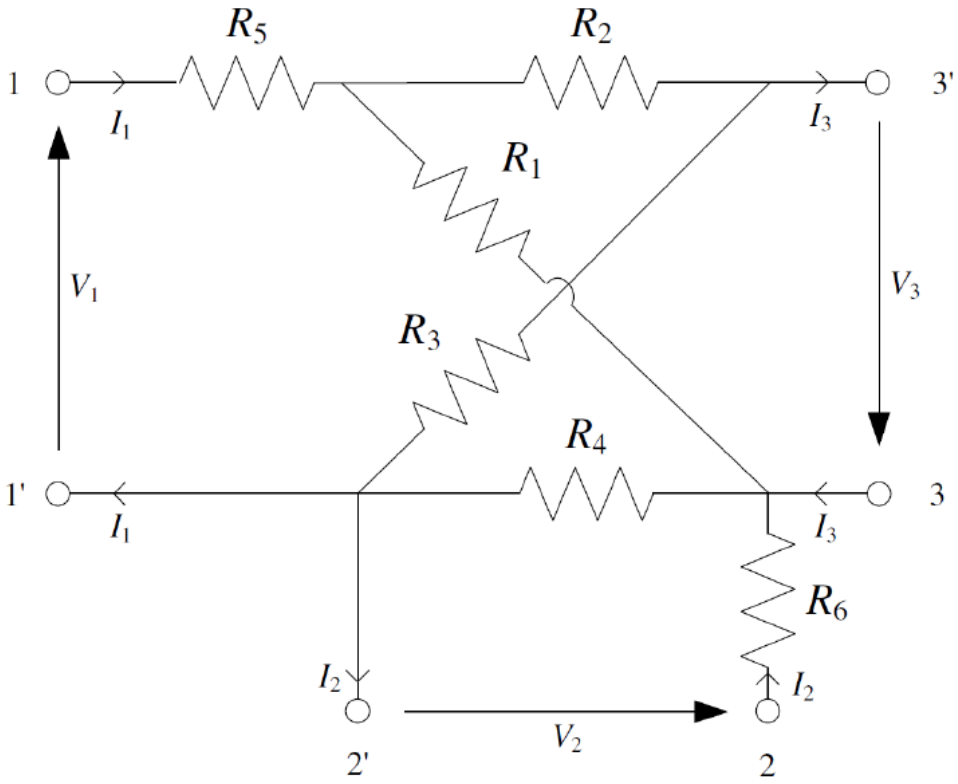
$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \text{ is paramount}$$

$$\Longleftrightarrow \begin{cases} p_{11} \geq \max\{|p_{12}|, |p_{13}|\}, \\ p_{22} \geq \max\{|p_{12}|, |p_{23}|\}, \qquad p_{33} \geq \max\{|p_{13}|, |p_{23}|\}, \\ p_{11}p_{22} - p_{12}^2 \geq \max\{|p_{11}p_{23} - p_{12}p_{13}|, |p_{12}p_{23} - p_{13}p_{22}|\}, \\ p_{22}p_{33} - p_{23}^2 \geq \max\{|p_{12}p_{33} - p_{13}p_{23}|, |p_{12}p_{23} - p_{13}p_{22}|\}, \\ p_{11}p_{33} - p_{13}^2 \geq \max\{|p_{11}p_{23} - p_{13}p_{12}|, |p_{12}p_{33} - p_{23}p_{13}|\}. \end{cases}$$

P. Slepian and L. Weinberg, "Synthesis applications of paramount and dominant matrices," in *Proceedings of Nat. Elec. Conf*, vol. 14, pp. 611–630, 1958.

Realizability of Three-Port Resistive Networks

Any 3×3 paramount matrix R is realizable by



after a finite number of cross-sign changes and a proper rearrangement of rows and columns.

B. D. H. Tellegen, Théorie der Wisselstromen, pp. 166–168. P. Noordhoff, 1952.

M. Z. Q. Chen, "Passive Network Synthesis of Restricted Complexity," Ph.D. Dissertation, Cambridge University Engineering Department, U.K., Aug. 2007.

Realizability For $n \leq 3$

Theorem

Paramountcy is a necessary and sufficient condition for any second- or third-order real symmetric matrix to be realizable as the impedance (or admittance) of a two- or three-port network.

Then case of $n > 3$ remains unsolved.

Dominant Matrices

Definition (dominant matrix)

A *dominant matrix* is a real symmetric matrix such that each of its main diagonal entries is not less than the sum of the absolute values of all other entries in the same row.

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1,n} \\ y_{12} & y_{22} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n} & y_{2,n} & \cdots & y_{n,n} \end{bmatrix} \text{ is a dominant matrix}$$

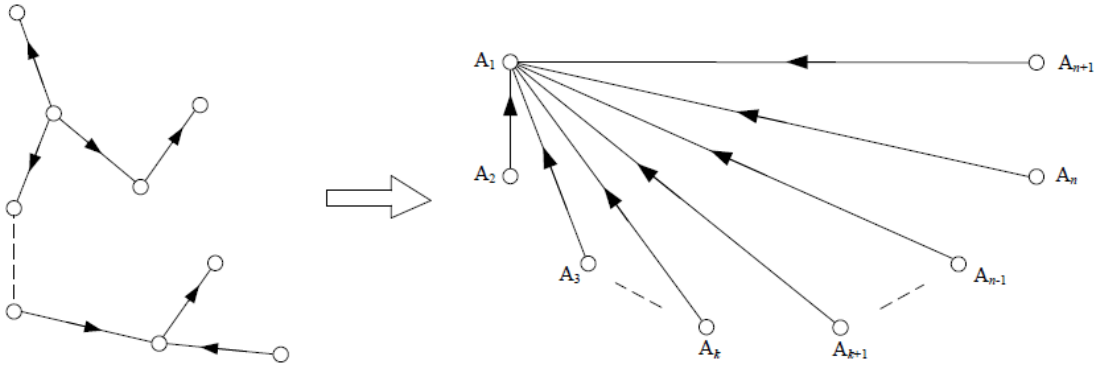
$$\Longleftrightarrow \qquad y_{ii} \geq \sum_{j=1,\ j\neq i}^{n} |y_{ij}| \quad \text{for all } i$$

I. Cederbaum, "Paramount matrices and realization of resistive 3-port networks," *Proceedings of the IEE-Part C: Monographs*, vol. 110, no. 11, pp. 1960–1964, 1963.

Realizability Conditions with $(n+1)$ Terminals

Theorem

A matrix $Y_n \in \mathbb{S}^{n\times n}$ is realizable as the admittance of a resistive n-port network containing $(n+1)$ terminals whose directed port graph is $\mathcal{G}_{pt_i}$, if and only if ***there exists $\mathfrak{W}_{iL}$ such that $\mathfrak{W}_{iL}\, Y_n \mathfrak{W}_{iL}^{\mathrm{T}}$ is a dominant matrix with all the off-diagonal entries being non-positive.***



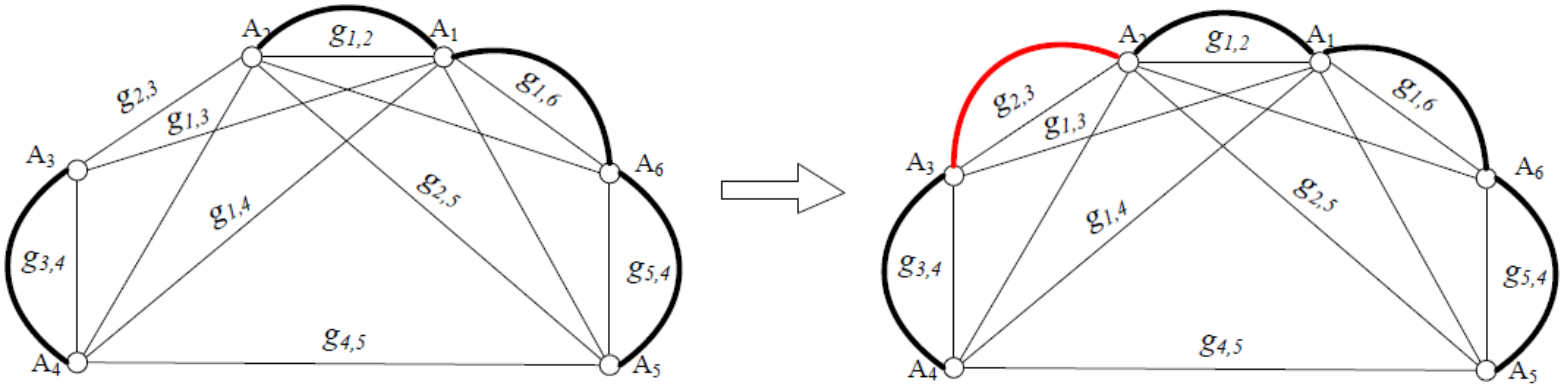
Port graph

Lagrangian tree

D. P. Brown and M. B. Reed, Necessary and sufficient conditions for *R*-graph synthesis, *Journal of the Franklin Institute*, vol. 273, no. 6, pp. 472–481, 1962.

Port Expansion

Expand an n-port resistive network with $(n+p)$ terminals to $(n'+1)$-terminal case where $n' = n+p-1$, by adding $(p-1)$ new ports.



Ports are made part of a tree
(4 ports, 6 terminals)

Ports constitute a tree
(5 ports, 6 terminals)

The realizability is converted into the known result.

E. A. Guillemin, On the realization of an nth-order G matrices, *IRE Trans. Circuit Theory*, vol. 8, no. 3, pp. 318–323, 1961.

Realizability Conditions of $(n+2)$-Terminal Case

Theorem

*A matrix $Y_n \in \mathbb{S}^n$ is realizable by an n-**port resistive network containing** $(n+2)$ **terminals**, if and only if after a proper rearrangement of rows and columns, there exist $\mathfrak{W} = \mathfrak{W}_{1P_1} \dot{+} \mathfrak{W}_{2P_2}$ and a set of parameters $p_1, p_2, \ldots, p_n, p$, such that the entries of $\mathfrak{W} Y_n \mathfrak{W}^{\mathrm{T}}$ satisfy $(y_{a,b} - y_{a,b+1}) - (y_{a-1,b} - y_{a-1,b+1}) \geq (p_a - p_{a-1})(p_{b+1} - p_b) \geq 0$ for $a, b = 1, 2, \ldots, (m-2)$ and $b \geq a$; $(y_{a,m-1} - y_{a-1,m-1}) \geq (p_a - p_{a-1})(p - p_{m-1}) \geq 0$ for $a = 1, 2, \ldots, (m-1)$; $(y_{i,j} - y_{i,j+1}) - (y_{i-1,j} - y_{i-1,j+1}) \geq (p_{i-1} - p_i)(p_j - p_{j+1}) \geq 0$ for $i, j = (m+1), (m+2), \ldots, n$ and $j \geq i$; $(y_{m,j} - y_{m,j+1}) \geq (p - p_m)(p_j - p_{j+1}) \geq 0$ for $j = m, (m+1), \ldots, n$; $(p_a - p_{a-1})(p_j - p_{j+1}) \geq -(y_{a,j} - y_{a,j+1}) + (y_{a-1,j} - y_{a-1,j+1})$ for $a = 1, 2, \ldots, (m-1)$ and $j = m, (m+1), \ldots, n$; $(p_a - p_{a-1})(p - p_m) \geq (y_{a,m} - y_{a-1,m})$ for $a = 1, 2, \ldots, (m-1)$; $(p - p_{m-1})(p - p_m) \geq -y_{m-1,m}$ and $(p - p_{m-1})(p_j - p_{j+1}) \geq (y_{m-1,j} - y_{m-1,j+1})$ for $j = m, (m+1), \ldots, n$. Here subscripts m and n are utilized to index the rows or columns in positions $m-1$ or less, and i and j are used for the rows or columns above $m-1$.*

K. R. Swaminathan and I. T. Frisch, Necessary conditions for the realizability of n-port resistive networks with more than $n+1$ nodes, *IEEE Trans. Circuit Theory*, vol. 12, no. 4, pp. 520–527, 1965.

Further Investigations on $(n+p)$-Terminal Case

1. Some new terms such as the *modified cut-set matrix* and the *K matrix* were utilized.
2. Most of works were still based on the port expansion method discussed by Guillemin, Swaminathan, *et al*.
3. The realizability problem of n-port resistive networks with $(n+p)$ terminals ($p \geq 2$) has not been solved.

F. J. Lupo and C. C. Halkias, Synthesis of n-port networks on two-tree port-structures, *IEEE Trans. Circuit Theory*, vol. 12, no. 4, pp. 571–577, 1965.

F. J. Lupo, The synthesis of transformerless n-port networks on multitree port structures, *IEEE Trans. Circuit Theory*, vol. 15, no. 3, pp. 211–220, 1968.

C. G. Jambotkar and Y. Tokad, Realization of nth-order matrices with two-tree port structures, Journal of the Franklin Institute, vol. 288, no. 4, pp. 245–260, 1969.

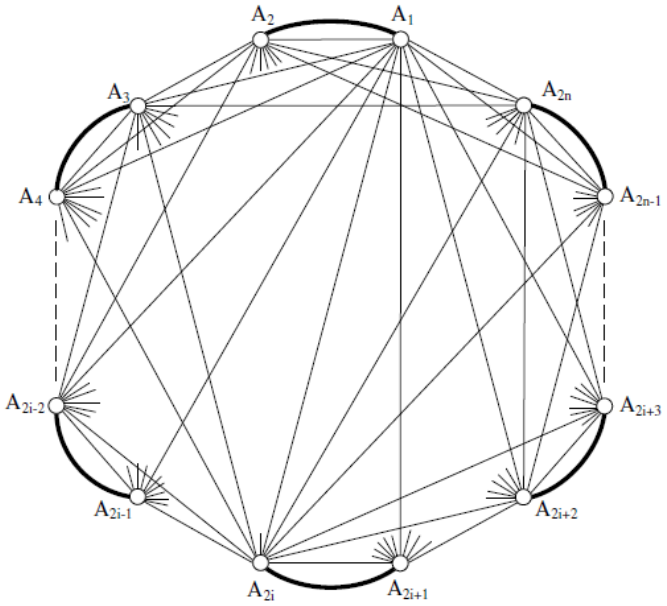
P. S. Reddy and K. Thulasiraman, "Synthesis of $(n+2)$-node resistive n-port networks," *IEEE Trans. Circuit Theory*, vol. 19, no. 1, pp. 20–24, 1972.

M. G. G. Naidu, P. S. Reddy, and K. Thulasiraman, $(n+2)$-node resistive n-port realizability of Y-matrices, *IEEE Trans. Circuit and Systems*, vol. 23, no. 5, pp. 254–260, 1976.

n-Port Resistive Networks Containing $2n$ Terminals

The n-port resistive networks with $2n$ terminals is an essential class of resistive networks:

- The symmetric topology may simplify realizability conditions.
- They contain the largest number of terminals.
- Contribute to realizability of more than $(n+1)$ terminals.



Realizability Conditions for $2n$-Terminal Case

Theorem A

A symmetric matrix $Y = [y_{ij}]_{n\times n}$ is realizable as an n-port resistive network N with $2n$ terminals, if and only if there exists a real $(2n-1)\times(n-1)$ matrix P in the form of

$$P = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \end{bmatrix}$$

with $\gamma_k = [p_1^{(k)},\ p_{n+1}^{(k)},\ \cdots, p_k^{(k)},\ p_{n+k}^{(k)},\ p_{k+1}^{(k)},\ 0,\ \cdots,\ p_{n-1}^{(k)},\ 0,\ p_n^{(k)}]^T$ for $1 \le k \le n-1$, such that

$$\begin{aligned} \max\{ &-(\alpha_i-\beta_{i-1})^T(\alpha_j-\beta_j),\ -(\alpha_i-\beta_i)^T(\alpha_j-\beta_{j-1})\} \le y_{ij} \\ &\le \min\{-(\alpha_i-\beta_{i-1})^T(\alpha_j-\beta_{j-1}),\ -(\alpha_i-\beta_i)^T(\alpha_j-\beta_j)\} \end{aligned}$$

for $1 \le i < j \le n$ and

$$y_{ii} \ge -(\alpha_i-\beta_{i-1})^T(\alpha_i-\beta_i), \qquad 1 \le i = j \le n,$$

where α_l^T is the $(2l-1)$th row of P with $1 \le l \le n$, β_m^T is the $2m$th row of P with $1 \le m \le n-1$, and $\beta_0 = \beta_n := 0_{(n-1)\times 1}$.

M. Z. Q. Chen *et al.*, "Synthesis of n-port resistive networks containing $2n$ terminals," *International Journal of Circuit Theory and Applications*, (DOI: 10.1002/cta.1951).

Values of the Elements

Theorem B

If a symmetric matrix $Y = [y_{ij}]_{n\times n}$ is realizable as the admittance of an n-port resistive network with 2n terminals, then the conductances of the resistors of the edges are given by

$$g_{2r-1,2s} = y_{r,s} + (\alpha_r - \beta_{r-1})^{\mathrm{T}}(\alpha_s - \beta_s),$$

for $1 \le r \le s \le n$, *and*

$$\begin{aligned} g_{2r-1,2s-1} &= -y_{r,s} - (\alpha_r - \beta_{r-1})^{\mathrm{T}}(\alpha_s - \beta_{s-1}), \\ g_{2r,2s-1} &= y_{r,s} + (\alpha_r - \beta_r)^{\mathrm{T}}(\alpha_s - \beta_{s-1}), \\ g_{2r,2s} &= -y_{r,s} - (\alpha_r - \beta_r)^{\mathrm{T}}(\alpha_s - \beta_s), \end{aligned}$$

for $1 \le r < s \le n$, *where* α_k *and* β_k *are obtained from the parameter matrix P as defined in Theorem A, and* $g_{h,l}$ *denotes the conductance of the element connecting vertices* A_h *and* A_l *for* $1 \le h < l \le 2n$.

M. Z. Q. Chen *et al.*, "Synthesis of *n*-port resistive networks containing 2*n* terminals," *International Journal of Circuit Theory and Applications*, (DOI: 10.1002/cta.1951).

Existing Results

	Admittance	Impedance
$n \leq 3$	Completely solved (paramountcy) [1]	Completely solved (paramountcy) [1]
$n+1$ terminals (or circuits)	Solved (needs to determine $\mathfrak{W}_{iL}$) [2,3]	Partly solved (realizability conditions are not available) [4]
$n+p$ terminals (or circuits)	Partly solved [5]	Seldom investigated
$2n$ terminals (or circuits)	Based on a parameter matrix [6]	Not investigated

Realizability conditions of impedances and admittances are different

[1] B. D. H. Tellegen, Théorie der Wisselstromen, pp. 166–168. P. Noordhoff, 1952.

[2] D. P. Brown and Y. Tokad, "On the synthesis of *R* networks," *IRE Trans. Circuit Theory*, vol. 8, no. 1, pp. 31–39, 1961.

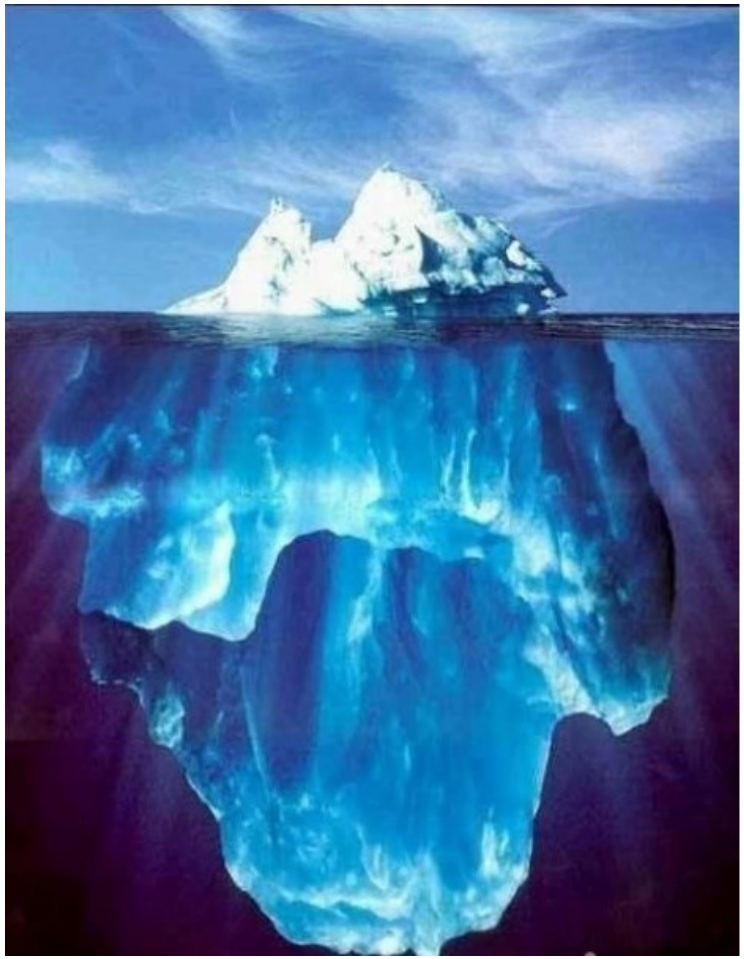
[3] D. P. Brown and M. B. Reed, Necessary and sufficient conditions for *R*-graph synthesis, *Journal of the Franklin Institute*, vol. 273, no. 6, pp. 472–481, 1962.

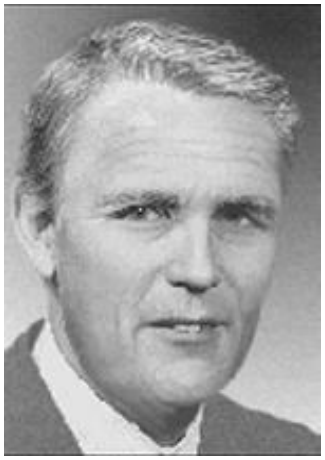
[4] I. Cederbaum, "Application of matrix algebra to network theory," *IRE Trans. Circuit Theory*, vol. 6, no. 5, pp. 127–137, 1959.

[5] K. R. Swaminathan and I. T. Frisch, Necessary conditions for the realizability of *n*-port resistive networks with more than $n+1$ nodes, *IEEE Trans. Circuit Theory*, vol. 12, no. 4, pp. 520–527, 1965.

[6] M. Z. Q. Chen *et al.*, "Synthesis of *n*-port resistive networks containing $2n$ terminals," *International Journal of Circuit Theory and Applications*, (DOI: 10.1002/cta.1951).

Resistive n-port Problem – The Worlds Greatest Career Destroyer





Rudolf Kalman, "Old and new directions of research in system theory," in *Perspectives in Mathematical System Theory, Control, and Signal Processing.* New York: Springer, 2010, vol. 398, LNCIS, pp. 3–13.

Abstract. The problem of classical **electrical network synthesis** (flourished between 1920–1970) is subjected to scientific critique. Conclusions: the first attacks on the problem were frustrated and eventually defeated by a naive over-reliance on engineering/physical intuition and shoving the mathematical issues whenever possible under the rug; now, by concentrating on essential mathematics, much of it known since the 19th century, *research* ***will be revived with spectacular prospects of scientific progress***.

The 2nd Workshop on Mathematical Aspects of Network Synthesis
26-27 September 2011

Conclusions

- A new mechanical element called the “inerter” was introduced.
- The inerter allows classical electrical network synthesis to be mapped exactly onto mechanical networks.
- Applications of the inerter: vehicle suspension, motorcycle steering, vibration absorption etc.
- The problem of n-port resistive network synthesis remains unresolved.

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Thank you for your attention!

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